
Answers to Exercises

Section 1.1

A1) The commutative property of the dot product is obvious from the definition, and the anti-commutative property of the cross product follows since unit vector $\hat{\mathbf{e}}$ for $\mathbf{b} \times \mathbf{a}$ is directed opposite to $\hat{\mathbf{e}}$ for $\mathbf{a} \times \mathbf{b}$.

The scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$ is the volume of the parallelepiped, or twice the volume of the tetrahedron, with sides defined by the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

A2) Addition of n -tuples is usually defined to be

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n) ,$$

and the multiplication operation by

$$\alpha(a_1, a_2, \dots, a_n) = (\alpha a_1, \alpha a_2, \dots, \alpha a_n) ,$$

$\forall \mathbf{a} \equiv (a_1, a_2, \dots, a_n), \mathbf{b} \equiv (b_1, b_2, \dots, b_n) \in V$ and $\forall \alpha \in K$.

The operations for matrices are of course quite analogous, with n -tuples being a special case (matrices with just one row or column). The axioms are all standard, in the usual matrix algebra.

Section 1.2

$$\begin{aligned} \text{A1) } \forall i \quad [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]_i &= \epsilon_{ijk} [\mathbf{a} \times \mathbf{b}]_j c_k = \epsilon_{ijk} \epsilon_{jlm} a_l b_m c_k = \epsilon_{kij} \epsilon_{jlm} a_l b_m c_k \\ &= (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) a_l b_m c_k = a_k c_k b_i - b_k c_k a_i = [\mathbf{a} \cdot \mathbf{c} \mathbf{b} - \mathbf{b} \cdot \mathbf{c} \mathbf{a}]_i . \end{aligned}$$

A2) From (1.6) and the symmetry of the orthogonality property (1.7),

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} , \quad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} , \quad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} ,$$

so consider say

$$1 = \mathbf{e}^1 \cdot \mathbf{e}_1 = \frac{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}{J \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} = \frac{1}{J \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3} \quad \text{because}$$

$$(\mathbf{e}_2 \times \mathbf{e}_3) \cdot (\mathbf{e}^2 \times \mathbf{e}^3) = \mathbf{e}_2 \cdot \mathbf{e}_3 \times (\mathbf{e}^2 \times \mathbf{e}^3) = \mathbf{e}_2 \cdot (\mathbf{e}_3 \cdot \mathbf{e}^3 \mathbf{e}^2 - \mathbf{e}_3 \cdot \mathbf{e}^2 \mathbf{e}^3) = 1.$$

Section 1.3

$$\begin{aligned} [\mathbf{A} + \mathbf{A}^T]^T &= \mathbf{A}^T + \mathbf{A} = [\mathbf{A} + \mathbf{A}^T] && \text{(symmetric)} ; \\ [\mathbf{A} - \mathbf{A}^T]^T &= \mathbf{A}^T - \mathbf{A} = -[\mathbf{A} - \mathbf{A}^T] && \text{(antisymmetric)} . \end{aligned}$$

Section 1.4

A1) (i) Let $\mathbf{v}$ be an arbitrary vector represented in the two forms $\mathbf{v} = v_j \mathbf{e}^j = v^j \mathbf{e}_j$. From the definitions of $\mathbf{e}_i$ and $\mathbf{e}^i$, we have $v_i = \mathbf{e}_i \cdot \mathbf{v}$ and $v^i = \mathbf{e}^i \cdot \mathbf{v}$. Thus

$$\left. \begin{aligned} (\mathbf{e}^i \mathbf{e}_i) \cdot \mathbf{v} &= v_i \mathbf{e}^i = \mathbf{v} \\ \mathbf{v} \cdot (\mathbf{e}^i \mathbf{e}_i) &= v^i \mathbf{e}_i = \mathbf{v} \end{aligned} \right\} \Rightarrow \mathbf{e}^i \mathbf{e}_i = \mathbf{I} , \text{ and}$$

$$\left. \begin{aligned} (\mathbf{e}_i \mathbf{e}^i) \cdot \mathbf{v} &= v^i \mathbf{e}_i = \mathbf{v} \\ \mathbf{v} \cdot (\mathbf{e}_i \mathbf{e}^i) &= v_i \mathbf{e}^i = \mathbf{v} \end{aligned} \right\} \Rightarrow \mathbf{e}_i \mathbf{e}^i = \mathbf{I} .$$

(The second result may also be deduced from the first, since $\mathbf{I}$ is symmetric.)

(ii) Pre and post dot $\mathbf{A}$ with the representations of $\mathbf{I}$ from (i), to exploit the identity $\mathbf{A} = \mathbf{I} \cdot \mathbf{A} \cdot \mathbf{I}$ — thus

$$\mathbf{A} = (\mathbf{e}_i \mathbf{e}^i) \cdot \mathbf{A} \cdot (\mathbf{e}_j \mathbf{e}^j) = \mathbf{e}_i A^i_j \mathbf{e}^j = A^i_j \mathbf{e}_i \mathbf{e}^j .$$

This trick of “inserting the dyadic identity” is often useful!

(iii) Use the identity

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{I} \cdot \mathbf{A} \cdot \mathbf{I} \cdot \mathbf{B} \cdot \mathbf{I}$$

with the representation $\mathbf{I} = \mathbf{e}_i \mathbf{e}^i$, yielding $\mathbf{e}_i \mathbf{e}^i \cdot \mathbf{A} \cdot \mathbf{e}_j \mathbf{e}^j \cdot \mathbf{B} \cdot \mathbf{e}_k \mathbf{e}^k = A^i_j B^j_k \mathbf{e}_i \mathbf{e}^k$ as required.

A2) We immediately have $\mathbf{e}_i \cdot \mathbf{I} \times \mathbf{a} \cdot \mathbf{e}_j = \mathbf{e}_i \times \mathbf{a} \cdot \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{a} \times \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{a} \times \mathbf{I} \cdot \mathbf{e}_j$.

A3) Change to a new basis set $\{\mathbf{e}'_i\}$ by using the identity $A^i_j = \mathbf{e}^i \cdot \mathbf{I} \cdot \mathbf{A} \cdot \mathbf{I} \cdot \mathbf{e}_j$ with $\mathbf{I} = \mathbf{e}'_k \mathbf{e}'^k$, so that

$$(0.1) \quad A^i_j = \mathbf{e}^i \cdot \mathbf{e}'_k A'^k_l \mathbf{e}'^l \cdot \mathbf{e}_j$$

where $A'^k_l \equiv \mathbf{e}'^k \cdot \mathbf{A} \cdot \mathbf{e}'_l$.

Define the transformation matrix

$$(0.2) \quad T^i_j \equiv \mathbf{e}'^i \cdot \mathbf{e}_j .$$

An interpretation of $\mathbf{e}^i \cdot \mathbf{e}'_j$ is also needed. Consider the matrix product

$$\begin{aligned} T^i_k \mathbf{e}^k \cdot \mathbf{e}'_j &= \mathbf{e}'^i \cdot \mathbf{e}_k \mathbf{e}^k \cdot \mathbf{e}'_j \\ &= \mathbf{e}'^i \cdot \mathbf{e}'_j \\ &= \delta_{ij} . \end{aligned}$$

Thus the $\mathbf{e}^i \cdot \mathbf{e}'_j$ are the components of the inverse transformation — i.e.

$$(0.3) \quad (T^{-1})^i_j = \mathbf{e}^i \cdot \mathbf{e}'_j ,$$

whence (0.1) may be written in the matrix form

$$A = T^{-1} A' T .$$

Invariance of the dyadic trace now follows from the invariance of the trace of a matrix product under cyclic permutation:

$$\text{Tr } A = \text{Tr } (T^{-1} A' T) = \text{Tr } (T T^{-1} A') = \text{Tr } A' .$$

Invariance of the determinant follows from the determinant of a matrix product being the product of determinants — i.e. $1 = \det(T^{-1} T) = \det T^{-1} \cdot \det T$, whence

$$\det A = \det T^{-1} \cdot \det A' \cdot \det T = \det A' .$$

Notes:

- $\text{Tr}(\mathbf{ab}) = \mathbf{a} \cdot \mathbf{e}^i \mathbf{e}_i \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$. Similarly, $\text{Tr}(\mathbf{ab} + \mathbf{cd}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{d}$ etc., and hence the trace of a dyadic is sometimes called its scalar.
- The double dot product of two dyadics is

$$(0.4) \quad \mathbf{A} : \mathbf{B} = \text{Tr}(\mathbf{A} \cdot \mathbf{B}) ,$$

hence we can write the trace as the double dot product

$$(0.5) \quad \text{Tr } \mathbf{A} = \mathbf{A} : \mathbf{I} = \mathbf{I} : \mathbf{A} .$$

- The determinant does not have as simple a coordinate-free representation as the trace, but the invariance property is useful for finding the determinant in specific cases because it allows us to choose a natural basis for the problem.

A4) It is convenient to choose the basis set $\{\mathbf{e}_1 = \mathbf{a}, \mathbf{e}_2 = \mathbf{b}, \mathbf{e}_3 = \mathbf{a} \times \mathbf{b}\}$. Then vector $\mathbf{a}$ has components $a^1 = \mathbf{e}^1 \cdot \mathbf{a} = \mathbf{e}^1 \cdot \mathbf{e}_1 = 1$, $a^2 = \mathbf{e}^2 \cdot \mathbf{a} = \mathbf{e}^2 \cdot \mathbf{e}_1 = 0$, $a^3 = \mathbf{e}^3 \cdot \mathbf{a} = \mathbf{e}^3 \cdot \mathbf{e}_1 = 0$; and the covariant components for the vector $\mathbf{b}$ are $b_1 = \mathbf{e}_1 \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b}$, $b_2 = \mathbf{e}_2 \cdot \mathbf{b} = |\mathbf{b}|^2$, $b_3 = \mathbf{e}_3 \cdot \mathbf{b} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{b} = 0$. Hence

$$(\mathbf{I} + \mathbf{ab})^i_j = \begin{bmatrix} 1 + \mathbf{a} \cdot \mathbf{b} & |\mathbf{b}|^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

so $\det(\mathbf{I} + \mathbf{ab}) = 1 + \mathbf{a} \cdot \mathbf{b}$.

A5) Recalling the definition of the transpose in Sect.1.3, for $\mathbf{A} = A_{ij}\mathbf{e}^i\mathbf{e}^j$ we have $\mathbf{A}^T = A_{ij}\mathbf{e}^j\mathbf{e}^i = A_{ji}\mathbf{e}^i\mathbf{e}^j$ (on exchanging the dummy indices), such that $(\mathbf{A}^T)_{ij} = A_{ji}$.

The other results sought are obtained in similar fashion.

A6) Dotting $\mathbf{e}_j dx^j = \mathbf{e}'_k dx'^k$ with the $\mathbf{e}^i$ (from the basis set reciprocal to $\{\mathbf{e}'_i\}$) yields

$$\mathbf{e}^i \cdot \mathbf{e}'_k dx'^k = dx'^i = \mathbf{e}^i \cdot \mathbf{e}_j dx^j ,$$

for comparison with

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j$$

from the coordinate transformation, identifying

$$\mathbf{e}^i \cdot \mathbf{e}_j = \frac{\partial x'^i}{\partial x^j} .$$

The transformation law for the vector field $\mathbf{v}(\mathbf{r})$ follows similarly, by dotting $v^j \mathbf{e}_j = v'^k \mathbf{e}'_k$ with $\mathbf{e}^i$. In passing, from (0.2) we also note that

$$T^i_j = \frac{\partial x'^i}{\partial x^j} .$$

A7) The dyadic dot product

$$\begin{aligned} A'^j{}_k B'^k{}_m &= \frac{\partial x'^j}{\partial x^p} \frac{\partial x^q}{\partial x'^k} A^p{}_q \frac{\partial x'^k}{\partial x^r} \frac{\partial x^s}{\partial x'^m} B^r{}_s = \frac{\partial x'^j}{\partial x^p} \frac{\partial x^q}{\partial x^r} \frac{\partial x^s}{\partial x'^m} A^p{}_q B^r{}_s \\ &= \frac{\partial x'^j}{\partial x^p} \frac{\partial x^s}{\partial x'^m} A^p{}_q B^q{}_s , \text{ since } \partial x^q / \partial x^r = \delta^q_r . \end{aligned}$$

Then dot product contraction, taking the dyadic dot to the double dot product, corresponds to setting $m = j$ in this result — i.e.

$$A'^j{}_k B'^k{}_j = \frac{\partial x'^j}{\partial x^p} \frac{\partial x^s}{\partial x'^j} A^p{}_q B^q{}_s = A^p{}_q B^q{}_p .$$

(The second order dyadic dot product is reduced to the zeroth order double dot product.)

Section 1.5

A1) (a) Applying the Cartesian form (1.35) for ∇ to $\mathbf{r}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ immediately yields $\nabla \mathbf{r} = \hat{\mathbf{i}}\hat{\mathbf{i}} + \hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}} = \mathbf{I}$; and contracting,

$$\nabla \cdot \mathbf{r} = \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} + \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 3$$

for the dot product, and $\nabla \times \mathbf{r} = \hat{\mathbf{i}} \times \hat{\mathbf{i}} + \hat{\mathbf{j}} \times \hat{\mathbf{j}} + \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0$ for the cross product.

(b) From (1.36) we have

$$\nabla \mathbf{r}(x^1, x^2, x^3) = \mathbf{e}^i \frac{\partial \mathbf{r}}{\partial x^i} = \mathbf{e}^i \mathbf{e}_i ,$$

recalling (1.5). The symmetry of $\mathbf{l}$ is evident in part (a), and appeared previously in (A1) of Sect.1.4 — i.e. $\mathbf{e}^i \mathbf{e}_i = \mathbf{e}_i \mathbf{e}^i$. At any point in space, from (1.5) the $\mathbf{e}_i$ are tangent whereas the $\mathbf{e}^i \equiv \nabla x^i$ are perpendicular to the level surfaces $x^i = \text{const}$ ($i \in \{1, 2, 3\}$), associated with the curvilinear coordinate system.

A2) We have

$$\nabla \mathbf{v} = \mathbf{e}^i \frac{\partial \mathbf{v}}{\partial x^i} = \frac{1}{J} \left(\mathbf{e}_2 \times \mathbf{e}_3 \frac{\partial \mathbf{v}}{\partial x^1} + \mathbf{e}_3 \times \mathbf{e}_1 \frac{\partial \mathbf{v}}{\partial x^2} + \mathbf{e}_1 \times \mathbf{e}_2 \frac{\partial \mathbf{v}}{\partial x^3} \right),$$

from (1.6) with $J \equiv \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3$. Consequently,

$$\nabla \mathbf{v} = \frac{1}{J} \left[\frac{\partial(\mathbf{e}_2 \times \mathbf{e}_3 \mathbf{v})}{\partial x^1} + \frac{\partial(\mathbf{e}_3 \times \mathbf{e}_1 \mathbf{v})}{\partial x^2} + \frac{\partial(\mathbf{e}_1 \times \mathbf{e}_2 \mathbf{v})}{\partial x^3} \right] = \frac{1}{J} \frac{\partial}{\partial x^i} (J \mathbf{e}^i \mathbf{v}),$$

on invoking the identity

$$\begin{aligned} & \frac{\partial(\mathbf{e}_2 \times \mathbf{e}_3)}{\partial x^1} + \frac{\partial(\mathbf{e}_3 \times \mathbf{e}_1)}{\partial x^2} + \frac{\partial(\mathbf{e}_1 \times \mathbf{e}_2)}{\partial x^3} \\ &= \frac{\partial}{\partial x^1} \left(\frac{\partial \mathbf{r}}{\partial x^2} \times \frac{\partial \mathbf{r}}{\partial x^3} \right) + \frac{\partial}{\partial x^2} \left(\frac{\partial \mathbf{r}}{\partial x^3} \times \frac{\partial \mathbf{r}}{\partial x^1} \right) + \frac{\partial}{\partial x^3} \left(\frac{\partial \mathbf{r}}{\partial x^1} \times \frac{\partial \mathbf{r}}{\partial x^2} \right) = 0. \end{aligned}$$

Contracting yields

$$(0.6) \quad \nabla \cdot \mathbf{v} = \frac{1}{J} \frac{\partial}{\partial x^i} (J \mathbf{e}^i \cdot \mathbf{v})$$

and $\nabla \times \mathbf{v}$ as

$$\begin{aligned} & \frac{1}{J} \left(\frac{\partial}{\partial x^1} [(\mathbf{e}_2 \times \mathbf{e}_3) \times \mathbf{v}] + \frac{\partial}{\partial x^2} [(\mathbf{e}_3 \times \mathbf{e}_1) \times \mathbf{v}] + \frac{\partial}{\partial x^3} [(\mathbf{e}_1 \times \mathbf{e}_2) \times \mathbf{v}] \right) \\ (0.7) \quad &= \frac{1}{J} \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ \mathbf{e}_1 \cdot \mathbf{v} & \mathbf{e}_2 \cdot \mathbf{v} & \mathbf{e}_3 \cdot \mathbf{v} \end{pmatrix}, \end{aligned}$$

by expanding the vector triple products to get

$$\begin{aligned} & \frac{\partial}{\partial x^1} [(\mathbf{e}_2 \cdot \mathbf{v} \mathbf{e}_3 - \mathbf{e}_2 \mathbf{e}_3 \cdot \mathbf{v})] + \frac{\partial}{\partial x^2} [(\mathbf{e}_3 \cdot \mathbf{v} \mathbf{e}_1 - \mathbf{e}_3 \mathbf{e}_1 \cdot \mathbf{v})] \\ &+ \frac{\partial}{\partial x^3} [(\mathbf{e}_1 \cdot \mathbf{v} \mathbf{e}_2 - \mathbf{e}_1 \mathbf{e}_2 \cdot \mathbf{v})] \\ &= \mathbf{e}_3 \frac{\partial}{\partial x^1} (\mathbf{e}_2 \cdot \mathbf{v}) - \mathbf{e}_2 \frac{\partial}{\partial x^1} (\mathbf{e}_3 \cdot \mathbf{v}) + \mathbf{e}_1 \frac{\partial}{\partial x^2} (\mathbf{e}_3 \cdot \mathbf{v}) \\ &- \mathbf{e}_3 \frac{\partial}{\partial x^2} (\mathbf{e}_1 \cdot \mathbf{v}) + \mathbf{e}_2 \frac{\partial}{\partial x^3} (\mathbf{e}_1 \cdot \mathbf{v}) - \mathbf{e}_1 \frac{\partial}{\partial x^3} (\mathbf{e}_2 \cdot \mathbf{v}) \end{aligned}$$

on noting that

$$\frac{\partial \mathbf{e}_3}{\partial x^1} - \frac{\partial \mathbf{e}_1}{\partial x^3} = \frac{\partial^2 \mathbf{r}}{\partial x^1 \partial x^3} - \frac{\partial^2 \mathbf{r}}{\partial x^3 \partial x^1} = 0, \text{ etc. .}$$

In passing, we mention that (1.42), (0.6) and (0.7) remain valid even when $\mathbf{v}$ is replaced by a higher order tensor field!

A3)

$$\begin{aligned} & (\nabla \psi) \cdot \nabla \times (\boldsymbol{\xi} \times \mathbf{B}) \\ &= -\nabla \cdot [(\nabla \psi) \times (\boldsymbol{\xi} \times \mathbf{B})] + (\nabla \times \nabla \psi) \cdot (\boldsymbol{\xi} \times \mathbf{B}) \quad (\text{from (1.39)}) \\ &= -\nabla \cdot [(\nabla \psi) \times (\boldsymbol{\xi} \times \mathbf{B})] \quad (\text{from (1.37)}) \\ &= \nabla \cdot (\mathbf{B} \boldsymbol{\xi} \cdot \nabla \psi) \quad (\text{since } \mathbf{B} \cdot \nabla \psi = 0) \\ &= \mathbf{B} \cdot \nabla (\boldsymbol{\xi} \cdot \nabla \psi) \quad (\text{since } \nabla \cdot \mathbf{B} = 0) . \end{aligned}$$

Section 1.6

A1) Taking partial derivatives of the position vector

$$\mathbf{r}(r, \theta, z) = r \cos \theta \hat{\mathbf{i}} + r \sin \theta \hat{\mathbf{j}} + z \hat{\mathbf{k}}$$

yields $\mathbf{e}_1 = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}} \equiv \hat{\mathbf{e}}_r$, $\mathbf{e}_2 = r(-\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}) \equiv r \hat{\mathbf{e}}_\theta$, and $\mathbf{e}_3 = \hat{\mathbf{k}} \equiv \hat{\mathbf{e}}_z$. The Jacobian $J \equiv \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = r$, so $\mathbf{e}^1 = \hat{\mathbf{e}}_r$, $\mathbf{e}^2 = \hat{\mathbf{e}}_\theta/r$ and $\mathbf{e}^3 = \hat{\mathbf{e}}_z$, hence the form for ∇ in cylindrical coordinates follows.

A2) The cylindrical coordinate representation is

$$\begin{aligned} & \nabla(v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_z \hat{\mathbf{e}}_z) \\ &= (\nabla v_r) \hat{\mathbf{e}}_r + (\nabla v_\theta) \hat{\mathbf{e}}_\theta + (\nabla v_z) \hat{\mathbf{e}}_z + v_r \nabla \hat{\mathbf{e}}_r + v_\theta \nabla \hat{\mathbf{e}}_\theta + v_z \nabla \hat{\mathbf{e}}_z \\ &= \frac{\partial v_r}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r + \frac{\partial v_r}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r + \frac{\partial v_\theta}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \frac{\partial v_\theta}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta \\ & \quad + \frac{\partial v_z}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z + \frac{\partial v_z}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z + \frac{v_r}{r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta - \frac{v_\theta}{r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r . \end{aligned}$$

Dot and cross contractions yield the cylindrical coordinate representations for $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$, as $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z\}$ is an orthonormal set — i.e.

$$\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_r = 0, \quad \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1,$$

$$\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_z = 0, \text{ and}$$

$$\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z, \quad \hat{\mathbf{e}}_\theta \times \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_r, \quad \hat{\mathbf{e}}_z \times \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta.$$

Note in passing that the representation for $\nabla \mathbf{v}$ could be written as a matrix

(on collecting terms with the same base dyad)

$$(0.8) \quad \begin{pmatrix} \frac{\partial v_r}{\partial r} & \frac{\partial v_\theta}{\partial r} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} & \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_r}{\partial z} & \frac{\partial v_\theta}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix},$$

and that $\nabla \cdot \mathbf{v} = \text{Tr} \nabla \mathbf{v}$ and $\nabla \times \mathbf{v}$ involve the diagonal and off-diagonal matrix elements, respectively.

$$\begin{aligned} A3) \quad \nabla \cdot \mathbf{T} &= \nabla \cdot (T_{rr} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + T_{r\theta} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta + T_{rz} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z + T_{\theta r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r + T_{\theta\theta} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta \\ &\quad + T_{\theta z} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z + T_{zr} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r + T_{zz} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z) \\ &= \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r \cdot \nabla T_{rr} + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r \cdot \nabla T_{r\theta} + \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r \cdot \nabla T_{rz} + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta \cdot \nabla T_{\theta r} + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta \cdot \nabla T_{\theta\theta} \\ &\quad + \hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta \cdot \nabla T_{\theta z} + \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z \cdot \nabla T_{zr} + \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z \cdot \nabla T_{z\theta} + \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z \cdot \nabla T_{zz} \\ &\quad + T_{rr} \nabla \cdot (\hat{\mathbf{e}}_r \hat{\mathbf{e}}_r) + T_{r\theta} \nabla \cdot (\hat{\mathbf{e}}_r \hat{\mathbf{e}}_\theta) + T_{rz} \nabla \cdot (\hat{\mathbf{e}}_r \hat{\mathbf{e}}_z) + T_{\theta r} \nabla \cdot (\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_r) \\ &\quad + T_{\theta\theta} \nabla \cdot (\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta) + T_{\theta z} \nabla \cdot (\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_z) + T_{zr} \nabla \cdot (\hat{\mathbf{e}}_z \hat{\mathbf{e}}_r) + T_{z\theta} \nabla \cdot (\hat{\mathbf{e}}_z \hat{\mathbf{e}}_\theta) + T_{zz} \nabla \cdot (\hat{\mathbf{e}}_z \hat{\mathbf{e}}_z) \\ &= \hat{\mathbf{e}}_r \frac{\partial T_{rr}}{\partial r} + \hat{\mathbf{e}}_\theta \frac{\partial T_{r\theta}}{\partial r} + \hat{\mathbf{e}}_z \frac{\partial T_{rz}}{\partial r} + \hat{\mathbf{e}}_r \frac{1}{r} \frac{\partial T_{\theta r}}{\partial \theta} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \hat{\mathbf{e}}_z \frac{1}{r} \frac{\partial T_{\theta z}}{\partial \theta} \\ &\quad + \hat{\mathbf{e}}_r \frac{\partial T_{zr}}{\partial z} + \hat{\mathbf{e}}_\theta \frac{\partial T_{z\theta}}{\partial z} + \hat{\mathbf{e}}_z \frac{\partial T_{zz}}{\partial z} + \frac{T_{rr}}{r} \hat{\mathbf{e}}_r + \frac{T_{r\theta}}{r} \hat{\mathbf{e}}_\theta + \frac{T_{\theta z}}{r} \hat{\mathbf{e}}_z + \frac{T_{\theta r}}{r} \hat{\mathbf{e}}_\theta - \frac{T_{\theta\theta}}{r} \hat{\mathbf{e}}_r, \end{aligned}$$

on recalling nonzero projections ($\hat{\mathbf{e}}_\theta \cdot \nabla \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta / r$, $\hat{\mathbf{e}}_\theta \cdot \nabla \hat{\mathbf{e}}_\theta = -\hat{\mathbf{e}}_r / r$) and the only nonzero contraction $\nabla \cdot \hat{\mathbf{e}}_r = 1/r$ originating from (1.44)-(1.46), to produce the last five (curvature) terms. Therefore in cylindrical coordinates

$$\begin{aligned} \nabla \cdot \mathbf{T} &= \left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta r}}{\partial \theta} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr}}{r} - \frac{T_{\theta\theta}}{r} \right) \hat{\mathbf{e}}_r \\ &\quad + \left(\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{z\theta}}{\partial z} + \frac{T_{r\theta}}{r} + \frac{T_{\theta r}}{r} \right) \hat{\mathbf{e}}_\theta \\ &\quad + \left(\frac{\partial T_{rz}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{\theta z}}{r} \right) \hat{\mathbf{e}}_z. \end{aligned}$$

The reader might verify that (0.7), with $\mathbf{T}$ replacing $\mathbf{v}$, reproduces this result! Since $\nabla^2 \mathbf{v} \equiv (\nabla \cdot \nabla) \mathbf{v} = \nabla \cdot (\nabla \mathbf{v})$, its cylindrical coordinate representation then follows directly by setting $\mathbf{T} = \nabla \mathbf{v}$. One might also use the identity $\nabla^2 \mathbf{v} = -\nabla \times (\nabla \times \mathbf{v}) + \nabla \nabla \cdot \mathbf{v}$ and invoke (1.43), (1.47) and (1.48) to obtain the expression for $\nabla^2 \mathbf{v}$ — viz.

$$\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \hat{\mathbf{e}}_r + \left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \hat{\mathbf{e}}_\theta + \nabla^2 v_z \hat{\mathbf{e}}_z,$$

$$\text{where} \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

A4) The position vector $\mathbf{r} = r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$ yields basis vectors

$$\begin{aligned}\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial r} = \sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}} \equiv \hat{\mathbf{e}}_r, \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial \theta} = r(\cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}}) \equiv r \hat{\mathbf{e}}_\theta, \\ \mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial \phi} = r \sin \theta (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) \equiv r \sin \theta \hat{\mathbf{e}}_\phi,\end{aligned}$$

where $\{\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi\}$ constitute an orthonormal set. Thus the Jacobian is

$$J \equiv \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = r^2 \sin \theta,$$

and from (1.6) we have the reciprocal set

$$\{\mathbf{e}^1 = \hat{\mathbf{e}}_r, \mathbf{e}^2 = \hat{\mathbf{e}}_\theta/r, \mathbf{e}^3 = \hat{\mathbf{e}}_\phi/(r \sin \theta)\},$$

hence

$$\nabla = \hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.$$

With the vector field representation $\mathbf{v} = v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta + v_\phi \hat{\mathbf{e}}_\phi$, the divergence $\nabla \cdot \mathbf{v}$ in spherical coordinates is

$$\hat{\mathbf{e}}_r \cdot \nabla v_r + \hat{\mathbf{e}}_\theta \cdot \nabla v_\theta + \hat{\mathbf{e}}_\phi \cdot \nabla v_\phi + v_r \nabla \cdot \hat{\mathbf{e}}_r + v_\theta \nabla \cdot \hat{\mathbf{e}}_\theta + v_\phi \nabla \cdot \hat{\mathbf{e}}_\phi.$$

Now

$$\begin{aligned}\nabla \cdot \hat{\mathbf{e}}_r &= \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot (\sin \theta \cos \phi \hat{\mathbf{i}} + \sin \theta \sin \phi \hat{\mathbf{j}} + \cos \theta \hat{\mathbf{k}}) \\ &= \frac{1}{r} (\cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}}) \cdot (\cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}}) \\ &\quad + \frac{1}{r \sin \theta} (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) \cdot (-\sin \theta \sin \phi \hat{\mathbf{i}} + \sin \theta \cos \phi \hat{\mathbf{j}}) = \frac{1}{r}, \\ \nabla \cdot \hat{\mathbf{e}}_\theta &= \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot (\cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}}) \\ &= \frac{1}{r} (\cos \theta \cos \phi \hat{\mathbf{i}} + \cos \theta \sin \phi \hat{\mathbf{j}} - \sin \theta \hat{\mathbf{k}}) \cdot (-\sin \theta \cos \phi \hat{\mathbf{i}} - \sin \theta \sin \phi \hat{\mathbf{j}} - \cos \theta \hat{\mathbf{k}}) \\ &\quad + \frac{1}{r \sin \theta} (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) \cdot (-\cos \theta \sin \phi \hat{\mathbf{i}} + \cos \theta \cos \phi \hat{\mathbf{j}}) = \frac{\cot \theta}{r}, \\ \nabla \cdot \hat{\mathbf{e}}_\phi &= \left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\mathbf{e}}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \cdot (-\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}) = 0,\end{aligned}$$

$$\begin{aligned}
\text{hence } \quad \nabla \cdot \mathbf{v} &= \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + 2 \frac{v_r}{r} + \cot \theta \frac{v_\theta}{r} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}.
\end{aligned}$$

This result follows immediately from the dot contracted form (0.6) of (1.42), since $J = r^2 \sin \theta$ and

$$\mathbf{e}^1 \cdot \mathbf{v} = \hat{\mathbf{e}}_r \cdot \mathbf{v} = v_r, \quad \mathbf{e}^2 \cdot \mathbf{v} = \frac{\hat{\mathbf{e}}_\theta}{r} \cdot \mathbf{v} = \frac{v_\theta}{r}, \quad \mathbf{e}^3 \cdot \mathbf{v} = \frac{\hat{\mathbf{e}}_\phi}{r \sin \theta} \cdot \mathbf{v} = \frac{v_\phi}{r \sin \theta}.$$

A5) The form for $\nabla \cdot \mathbf{v}$ follows immediately from (0.6) with $\mathbf{v} = v^i \hat{\mathbf{e}}_i$, $J = h_1 h_2 h_3$ and $J \mathbf{e}^1 = \mathbf{e}_2 \times \mathbf{e}_3 = h_2 h_3 \hat{\mathbf{e}}_1$, etc.

Likewise, the form for $\nabla \times \mathbf{v}$ follows immediately from (0.7) with $\mathbf{v} = v^i \hat{\mathbf{e}}_i$ and $\mathbf{e}_1 = h_1 \hat{\mathbf{e}}_1$, etc.

The form for $\nabla^2 \phi = \nabla \cdot \nabla \phi$ of course follows from $\nabla \cdot \mathbf{v}$ with $\mathbf{v} \equiv \nabla \phi$, using ∇ given by (1.50).

Identifying $\{x^1 = r, x^2 = \theta, x^3 = z\}$ such that $\{h_1 = 1, h_2 = r, h_3 = 1\}$, and $\{\hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_z\}$ and $\{v^1 = v_r, v^2 = v_\theta, v^3 = v_z\}$ immediately yields (1.47)–(1.49).

Section 1.7

A1) The line integrals are

$$\oint_C \mathbf{r} \cdot d\mathbf{r} = \int_S d\mathbf{S} \times \nabla \cdot \mathbf{r} = \int_S d\mathbf{S} \cdot \nabla \times \mathbf{r} = 0 \quad \text{since } \nabla \times \mathbf{r} = 0, \text{ and}$$

$$\oint_C \mathbf{r} \times d\mathbf{r} = - \int_S (d\mathbf{S} \times \nabla) \times \mathbf{r} = - \int_S [(\nabla \mathbf{r}) \cdot d\mathbf{S} - d\mathbf{S} \nabla \cdot \mathbf{r}] = 2 \int_S d\mathbf{S}$$

since $(\nabla \mathbf{r}) \cdot d\mathbf{S} = \mathbf{l} \cdot d\mathbf{S} = dS$ and $\nabla \cdot \mathbf{r} = 3$.

A2) We have

$$\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \hat{\mathbf{k}}$$

and $d\mathbf{S} = \mathbf{r}_x \times \mathbf{r}_y dx dy = \hat{\mathbf{k}} dx dy$, since the position vector of any point on the hemisphere S is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$. Thus if the projection of S on the XOY-plane is denoted by A ,

$$\int_S \nabla \times \mathbf{v} \cdot d\mathbf{S} = \int_A dx dy = \pi;$$

and noting that $z = 0$ on C and using the plane polar coordinate θ , this equals

$$\begin{aligned}\oint_C (2x - y) \hat{\mathbf{i}} \cdot d\mathbf{r} &= \int_0^{2\pi} (2 \cos \theta - \sin \theta) (\hat{\mathbf{e}}_r \cos \theta - \hat{\mathbf{e}}_\theta \sin \theta) \cdot \hat{\mathbf{e}}_\theta d\theta \\ &= \int_0^{2\pi} (2 \cos \theta - \sin \theta) \sin \theta d\theta \\ &= \int_0^{2\pi} [\sin 2\theta + \frac{1}{2}(1 - \cos 2\theta)] d\theta = \pi .\end{aligned}$$

Section 1.8

A1) We have

$$\begin{aligned}\frac{1}{2} \oint_S d\mathbf{S} \times (\mathbf{a} \times \mathbf{r}) &= \frac{1}{2} \int_V \nabla \times (\mathbf{a} \times \mathbf{r}) d\tau \\ &= \frac{1}{2} \int_V [\mathbf{a} \nabla \cdot \mathbf{r} - \mathbf{a} \cdot \nabla \mathbf{r}] d\tau = \mathbf{a} \int_V d\tau = V \mathbf{a} ,\end{aligned}$$

since $\nabla \cdot \mathbf{r} = 3$ and $\mathbf{a} \cdot \nabla \mathbf{r} = \mathbf{a} \cdot \mathbf{l} = \mathbf{a}$ (constant vector).

A2) (a) The nonzero surface integrals are $-1/2$ on cube face $y = 0$, $1/2$ on $y = 1$, 1 on $x = 1$, $1/2$ on $z = 1$ yielding $3/2$ over the closed surface S . The volume integral is

$$\int_V \nabla \cdot \mathbf{F} d\tau = \int_0^1 dz \int_0^1 dy \int_0^1 (2x + y) dx = \frac{3}{2} .$$

(b) Cylindrical polar coordinates (r, θ, z) are convenient. The only nonzero surface integral contribution is on $z = h$ — viz.

$$\int_{r=0}^a \int_{\theta=0}^{2\pi} \mathbf{F} \cdot \hat{\mathbf{k}} r dr d\theta = \pi a^2 h^2 .$$

Since $\nabla \cdot \mathbf{F} = 2z$ and $J = r$ such that $d\tau = r dr d\theta dz$, the volume integral is

$$\int_V \nabla \cdot \mathbf{F} d\tau = \int_{z=0}^h \int_{\theta=0}^{2\pi} \int_{r=0}^a 2z r dr d\theta dz = \pi a^2 h^2 .$$

(c) Spherical polar coordinates (r, θ, ϕ) are convenient, shown in Fig.1.2. The surface integral on the base $z = 0$ is zero, and on the hemispherical surface $r = a$ the integral is

$$\begin{aligned}\int_S \mathbf{F} \cdot \frac{\mathbf{r}}{a} dS &= \int_S (2xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}) \cdot \frac{(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{a} dS \\ &= \int_S \frac{2x^2z + y^2z + z^3}{a} dS \\ &= a^4 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (\sin^2 \theta \cos^2 \phi + 1) \cos \theta \sin \theta d\theta d\phi = \frac{5}{4} \pi a^4 ,\end{aligned}$$

as $x = a \sin \theta \cos \phi$, $y = a \sin \theta \sin \phi$, $z = a \cos \theta$ on S and $dS = a^2 \sin \theta d\theta d\phi$. Since $\nabla \cdot \mathbf{F} = 5z = 5r \cos \theta$ and $J = r^2 \sin \theta$ such that $d\tau = r^2 \sin \theta dr d\theta d\phi$, the volume integral is

$$\int_V \nabla \cdot \mathbf{F} d\tau = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^a 5r^3 \cos \theta \sin \theta dr d\theta d\phi = \frac{5}{4} \pi a^4 .$$

A3) The results follow by invoking (1.59) and a Mean Value Theorem. For example, from the Divergence Theorem (1.60)

$$\int_{\delta S} d\mathbf{S} \cdot \mathbf{v} = \int_{\Delta V} \nabla \cdot \mathbf{v} d\tau = \overline{\nabla \cdot \mathbf{v}} \int_{\Delta V} d\tau = \overline{\nabla \cdot \mathbf{v}} \Delta V ,$$

where $\overline{\nabla \cdot \mathbf{v}}$ is some intermediate value between the maximum and minimum value attained by $\nabla \cdot \mathbf{v}$ within ΔV . Thus in the limit $\Delta V \rightarrow 0$ such that point P remains inside ΔV , the value $\overline{\nabla \cdot \mathbf{v}} \rightarrow \nabla \cdot \mathbf{v}$ at P and hence

$$\frac{\int_{\Delta S} \mathbf{v} \cdot d\mathbf{S}}{\Delta V} \rightarrow \nabla \cdot \mathbf{v} .$$

Likewise, the first result follows from

$$\int_{\Delta S} \phi d\mathbf{S} = \int_{\Delta V} \nabla \phi d\tau ,$$

and the third result from

$$\int_{\Delta S} d\mathbf{S} \times \mathbf{v} = \int_{\Delta V} \nabla \times \mathbf{v} d\tau .$$

Section 1.9

The Laplacian of the surface integral in (1.67)

$$\nabla_0^2 \oint_S \left(\frac{1}{|\mathbf{r} - \mathbf{r}_0|} \nabla \phi - \phi \nabla \frac{1}{|\mathbf{r} - \mathbf{r}_0|} \right) \cdot d\mathbf{S}$$

vanishes since $\nabla_0^2 (1/|\mathbf{r} - \mathbf{r}_0|) = 0$ and the operators ∇_0^2 and ∇ commute. Thus the Laplacian of (1.67) yields

$$\nabla_0^2 \phi = \nabla_0^2 \left(- \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau \right)$$

on substituting the Poisson equation $\nabla^2 \phi = 4\pi\rho$, such that a particular solution is

$$\phi(\mathbf{r}_0) = - \int_V \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}_0|} d\tau .$$

Since the Laplace equation is the homogeneous form of the linear Poisson equation, the general solution is a superposition of the linearly independent solutions of the Laplace equation and this particular solution.

Section 1.10

A1) The distribution corresponding to the function $f(x) = 2H(x) - 1$ is

$$\begin{aligned}\underline{f(x)} \phi &= \int_{-\infty}^{\infty} [2H(x-y) - 1] \phi(y) dy \\ &= 2 \int_{-\infty}^{\infty} H(x-y) \phi(y) dy - \int_{-\infty}^{\infty} \phi(y) dy ,\end{aligned}$$

hence

$$\begin{aligned}\underline{f'(x)} \phi &= \frac{d}{dx} [\underline{f(x)} \phi] \\ &= 2 \frac{d}{dx} \int_{-\infty}^{\infty} H(x-y) \phi(y) dy \\ &= 2 \frac{d}{dx} \int_{-\infty}^x \phi(y) dy \\ &= 2\phi(x) \\ &= 2\underline{\delta(x)} \phi \text{ --- i.e. } f'(x) = \delta(x) .\end{aligned}$$

A2) When $k = 1$, we have $xy = f(x)$ such that $y = x^{-1}f(x) + c_1\delta(x)$ since $x\delta(x) = 0$ follows from $f(x)\delta(x) = f(0)\delta(x)$.

When $k = 2$, we have $x^2y = f(x)$ such that $y = x^{-2}f(x) + c_1\delta(x) + c_2\delta'(x)$, as $x^2\delta(x) = 0$ follows from $f(x)\delta(x) = f(0)\delta(x)$ and $x^2\delta'(x) = 0$ follows from $f(x)\delta(x) = f(0)\delta'(x) - f'(0)\delta(x)$.

Now $x\delta'(x) = -\delta(x)$ also follows from $f(x)\delta(x) = f(0)\delta'(x) - f'(0)\delta(x)$, so on differentiating we have $x\delta''(x) + \delta'(x) = -\delta'(x) \Rightarrow x\delta''(x) = -2\delta'(x)$ such that $x^2\delta''(x) = -2x\delta'(x) = 2\delta(x)$ and hence $x^3\delta''(x) = 2x\delta(x) = 0$. Then differentiating $x^2\delta''(x) = -2x\delta'(x) = 2\delta(x)$ we therefore get $x^4\delta'''(x) = 0$, etc. so $x^k\delta^{(k-1)}(x) = 0 \forall k$ may be proven inductively, and hence the result.

A3) Since $H(x - x_0 - y)$ is 0 when $y > x - x_0$ and 1 when $y < x - x_0$,

$$\begin{aligned}\underline{H(x - x_0)f(x)} \phi &= \int_{-\infty}^{\infty} H(x - x_0 - y) f(x - y) \phi(y) dy \\ &= \int_{-\infty}^{x-x_0} f(x - y) \phi(y) dy ,\end{aligned}$$

so provided the derivative function $f'(x)$ exists we have

$$\begin{aligned}\frac{d}{dx} [\underline{H(x - x_0)f(x)} \phi] &= f(x_0)\phi(x - x_0) + \int_{-\infty}^{x-x_0} f'(x - y) \phi(y) dy \\ &= f(x_0) \underline{\delta(x - x_0)} \phi + \underline{H(x - x_0)f'(x)} \phi .\end{aligned}$$

On the other hand, we also have

$$\underline{\delta(x - x_0)f(x)} \phi = \frac{d}{dx} [\underline{H(x - x_0)f(x)} \phi] - \underline{H(x - x_0)f'(x)} \phi ,$$

so $\underline{\delta(x - x_0)f(x)} \phi = f(x_0) \underline{\delta(x - x_0)} \phi$ --- i.e. $\underline{\delta(x - x_0)f(x)} = f(x_0)\delta(x - x_0)$.

Consequently,

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx &= \int_{-\infty}^{\infty} f(x_0) \delta(x - x_0) dx \\
&= f(x_0) \int_{-\infty}^{\infty} \delta(x - x_0) dx \\
&= f(x_0) \int_{-\infty}^{\infty} \frac{d}{dx} H(x - x_0) dx \\
&= f(x_0) [H(x - x_0)]_{-\infty}^{\infty} = f(x_0) .
\end{aligned}$$

If $f''(x)$ exists we likewise have $\delta(x - x_0)f'(x) = f(x_0)\delta(x - x_0)$ in addition to $\delta(x - x_0)f(x) = f(x_0)\delta(x - x_0)$, and therefore

$$\begin{aligned}
\underline{\delta'(x - x_0)f(x) \phi} &= \frac{d}{dx} [\underline{\delta(x - x_0)f(x) \phi}] - \underline{\delta(x - x_0)f'(x) \phi} \\
&= f(x_0) \frac{d}{dx} [\underline{\delta(x - x_0) \phi}] - \underline{\delta(x - x_0)f'(x) \phi} \\
&= f(x_0) \underline{\delta'(x - x_0) \phi} - f'(x_0) \underline{\delta(x - x_0) \phi}
\end{aligned}$$

— i.e. we now have $\delta'(x - x_0)f(x) = f(x_0)\delta'(x - x_0) - f'(x_0)\delta(x - x_0)$ such that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \delta'(x - x_0)f(x) dx \\
&= \int_{-\infty}^{\infty} f(x_0)\delta(x - x_0) dx - \int_{-\infty}^{\infty} f'(x_0)\delta(x - x_0) dx \\
&= f(x_0) \int_{-\infty}^{\infty} \delta'(x - x_0) dx - f'(x_0) \int_{-\infty}^{\infty} \delta(x - x_0) dx \\
&= f(x_0) \int_{-\infty}^{\infty} \delta'(x - x_0) dx - f'(x_0) \int_{-\infty}^{\infty} \frac{d}{dx} H(x - x_0) dx \\
&= -f'(x_0) [H(x - x_0)]_{-\infty}^{\infty} = -f'(x_0) ,
\end{aligned}$$

on assuming that the integral involving the delta function derivative is zero.

A4) First note that boundary conditions only apply to corresponding classical solutions.

(a) On integrating, we have a distribution equation corresponding to the classical equation $y(x) = H(x - 1) + c$, where the initial condition $y(0) = 0$ yields $H(-1) + c = 0 \Rightarrow c = 0$, so the solution is $y(x) = H(x - 1)$.

(b) We get the distribution equation $dy/dx = H(x - 1) + \delta(x - 1) + c$ on integrating; and since $d[xH(x - 1)]/dx = H(x - 1) + x\delta(x - 1) = H(x - 1) + \delta(x - 1)$, a second integration yields $y(x) = xH(x - 1) + c_1x + c_2$, again corresponding to a classical equation. The condition $y(0) = 0$ implies $c_2 = 0$, and since $y'(x) = c_1$ in the neighbourhood of $x = 0$ the condition $y'(0) = 0$ implies $c_1 = 0$, so the solution is $y(x) = xH(x - 1)$.

(c) The differential equation may be rewritten $x d^2(xy)/dx^2 = -x\delta'(x)$ since $x\delta'(x) = -\delta(x)$, so from (Q2) we have that $d^2(xy)/dx^2 = -\delta'(x) + c_1\delta(x)$. On integrating this result twice, we have $d(xy)/dx = -\delta(x) + c_1H(x) + c_2$ and then $xy(x) = -H(x) + c_1xH(x) + c_2x + c_3$, so again invoking (Q2) we obtain

$$y(x) = -\frac{H(x)}{x} + c_1H(x) + c_2 + \frac{c_3}{x} + c_4\delta(x) .$$

To render a corresponding classical equation, evidently we must set $c_4 = 0$. Then the boundary condition $y \rightarrow 0$ when $x \rightarrow \infty$ implies $c_2 = 0$ and the boundary condition $y' \rightarrow 0$ when $x \rightarrow -\infty$ implies $c_1 = 0$, so we obtain the solution $y(x) = [c_3 - H(x)]/x$, which still involves one arbitrary constant!

A5) Integrals such as $\int_{-\infty}^{\infty} e^{ixy} dy$ (familiar as the Fourier transform of unity) do not converge as functions of x but often do as distributions. Thus

$$\underline{e^{ixy}} \phi = \int_{-\infty}^{\infty} e^{iy(x-z)} \phi(z) dz = 2\pi\phi(x) = 2\pi\underline{\delta(x)} \phi$$

— i.e. $\int_{-\infty}^{\infty} e^{ixy} dy = 2\pi\delta(x)$. Differentiation with respect to x then renders $\int_{-\infty}^{\infty} e^{ixy} y dx = 2\pi(-i)\delta'(x)$, etc. — i.e.

$$(0.9) \quad \int_{-\infty}^{\infty} e^{ixy} y^n dy = 2\pi(-i)^n \delta^{(n)}(x) ,$$

on successive differentiation to any order n . The Fourier transform pair is

$$f(x) = \int_{-\infty}^{\infty} e^{ixy} \hat{f}(y) dy \quad \text{and} \quad \hat{f}(y) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixy} f(x) dx .$$

From the Taylor expansion $e^{-ixy} = \sum_{n=0}^{\infty} (1/n!)(-ixy)^n$, we have

$$\begin{aligned} \hat{f}(y) &= \sum_{n=0}^{\infty} \frac{1}{2\pi n!} \int_{-\infty}^{\infty} f(x) (-ixy)^n dx \\ &= \sum_{n=0}^{\infty} (-iy)^n f_n / (2\pi n!) \end{aligned}$$

where $f_n = \int_{-\infty}^{\infty} x^n f(x) dx$, such that

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (-iy)^n f_n / (2\pi n!) e^{ixy} dy \\ &= \sum_{n=0}^{\infty} (-1)^n f_n \delta^{(n)}(x) / n! \end{aligned}$$

on invoking (0.9).

Section 1.11

A1) The potential function $\phi(\mathbf{r})$ is to satisfy the Laplace equation under (1.71). From (1.76), for any point outside the conducting spherical surface

$$\nabla G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi} \left(\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} - \frac{a(\mathbf{r} - \mathbf{r}'_0)}{|\mathbf{r}_0||\mathbf{r} - \mathbf{r}'_0|^3} \right),$$

since $\nabla|\mathbf{r} - \mathbf{r}_0|$ is

$$\left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}$$

and likewise $\nabla|\mathbf{r} - \mathbf{r}'_0| = (\mathbf{r} - \mathbf{r}'_0)/|\mathbf{r} - \mathbf{r}'_0|$. Noting that the unit normal is $\hat{\mathbf{n}} = -\mathbf{r}/a$ on the spherical surface S of radius a , the integrand on the right-hand side of the relevant equation (1.72) involves the factor

$$\begin{aligned} [\nabla G(\mathbf{r}, \mathbf{r}_0)] \cdot d\mathbf{S} &= \frac{1}{4\pi} \left[\frac{(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3} \cdot \left(\frac{-\mathbf{r}}{a} \right) - \frac{a(\mathbf{r} - \mathbf{r}'_0)}{|\mathbf{r}_0||\mathbf{r} - \mathbf{r}'_0|^3} \cdot \left(\frac{-\mathbf{r}}{a} \right) \right] dS \\ &= \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}_0|^3} \left[-|\mathbf{r}| + \frac{\mathbf{r}_0 \cdot \mathbf{r}}{a} + \frac{|\mathbf{r}_0|^2}{a^2} \left(|\mathbf{r}| - \frac{\mathbf{r}'_0 \cdot \mathbf{r}}{a} \right) \right] dS \end{aligned}$$

since $|\mathbf{r} - \mathbf{r}'_0| = (a/|\mathbf{r}_0|)|\mathbf{r} - \mathbf{r}_0|$ — i.e. on $|\mathbf{r}| = a$ we have the factor

$$\frac{1}{4\pi a} \frac{1}{|\mathbf{r} - \mathbf{r}_0|^3} (|\mathbf{r}_0|^2 - a^2) dS + \frac{1}{4\pi a} \frac{1}{|\mathbf{r} - \mathbf{r}_0|^3} (\mathbf{r}_0 - \mathbf{r}'_0) \cdot \mathbf{r} dS.$$

The contribution to the integral from the second term is found to be zero (employing spherical coordinates where $\theta = 0$ coincides with the $\mathbf{r}_0 - \mathbf{r}'_0$ direction), and the contribution from the first term renders the result given. The derivation for any point inside the spherical surface similarly follows from the second part of (1.76).

A2) The Fourier transform of the differential equation for the Green function is

$$\left(-k^2 + \frac{\omega^2}{c^2} \right) \hat{G}(\mathbf{k}, \omega) = 1 \quad \Rightarrow \quad \hat{G}(\mathbf{k}, \omega) = \frac{c^2}{\omega^2 - k^2 c^2},$$

and hence $G(\mathbf{r}, t)$ on inversion as given. Thus the solution of the forced wave equation is

$$\phi(\mathbf{r}_0, t_0) = -\frac{c}{4\pi} \int \frac{\delta[|\mathbf{r} - \mathbf{r}_0| - c(t_0 - t)]}{|\mathbf{r} - \mathbf{r}_0|} f(\mathbf{r}, t) d\mathbf{r} dt$$

and hence the result, following integration with respect to t .

Section 2.2

A1) (a) If $\mathbf{v} = ky \hat{\mathbf{i}}$ (k an arbitrary constant), then $\nabla \mathbf{v} = k \hat{\mathbf{i}} \hat{\mathbf{j}}$ where $\hat{\mathbf{j}}$ is the Cartesian basis vector in the y -direction. Hence $\nabla \cdot \mathbf{v} = k \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = 0$ and $\mathbf{t} = -k\mu(\hat{\mathbf{i}} \hat{\mathbf{j}} + \hat{\mathbf{j}} \hat{\mathbf{i}})$, the viscous contribution to the pressure tensor for this incompressible shear flow. The corresponding viscous surface force on the fluid per unit area is $-\hat{\mathbf{n}} \cdot \mathbf{t}$. For example, if the fluid occupies the region $y > 0$, the force per unit area on the fluid from a fixed boundary at $y = 0$ would be $\hat{\mathbf{j}} \cdot \mathbf{t} = -k\mu \hat{\mathbf{i}}$, since the fluid outward normal $\hat{\mathbf{n}} = -\hat{\mathbf{j}}$. Thus the force on the fluid is in the negative x -direction, opposing the flow. (The equal and opposite viscous force acting on the boundary due to the flow is $\hat{\mathbf{n}} \cdot \mathbf{t} = -\hat{\mathbf{j}} \cdot \mathbf{t} = k\mu \hat{\mathbf{i}}$.)

(b) For the fluid velocity $\mathbf{v} = k\mathbf{r}/3$, $\nabla \mathbf{v} = k\mathbf{I}/3$ and $\nabla \cdot \mathbf{v} = k$ such that $\mathbf{p} = (p - \mu_v k)\mathbf{I}$. Thus the shear viscosity term is zero, whereas the volume viscosity (expansive friction) term produces a contribution to the pressure, opposing the fluid expansion when $k > 0$ (or its contraction when $k < 0$).

Section 2.3

A1) In Cartesian coordinates where the z -axis is directed vertically upward, the hydrostatic equilibrium equation for the unstratified incompressible fluid (with constant density ρ) in the region $z \leq 0$ is $\nabla p = -\rho g \hat{\mathbf{k}}$, where g denotes the constant gravitational acceleration. The buoyancy force upward due to the fluid pressure is thus

$$\int_S p d\mathbf{S} \cdot \hat{\mathbf{k}} = \int_V \hat{\mathbf{k}} \cdot \nabla p d\tau = \rho g \times \text{immersed volume},$$

where we conveniently take the reference pressure at the upper fluid surface $z = 0$ to be zero.

Section 2.4

A1) Consider the reference frame rotating with constant angular velocity $\boldsymbol{\omega}_0 = \omega_0 \hat{\mathbf{k}}$ about the vertical z -axis in which the fluid appears static, so from (2.29) we have

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}_0 \times (\boldsymbol{\omega}_0 \times \mathbf{r}) = -\omega_0^2 \mathbf{r}$$

for a position vector $\mathbf{r}$ restricted to the horizontal plane (so $\boldsymbol{\omega}_0 \cdot \mathbf{r} = 0$). Adopting cylindrical coordinates, the component of the equilibrium equation in the radial direction is $\partial p / \partial r = \rho r \omega_0^2$ (a “centrifugal force”), and in the axial direction we have $\partial p / \partial z = -\rho g$. Since $(1/r) \partial p / \partial \theta = 0$ is the

component in the azimuthal direction, the configuration is evidently axisymmetric. For $\rho = \text{const}$, integration yields

$$p = \frac{1}{2}\rho r^2 \omega_0^2 + g(z) , \quad p = -\rho g z + h(r)$$

where $g(z)$ and $h(r)$ are arbitrary functions, whence the pressure

$$p(r, z) = \frac{1}{2}\rho r^2 \omega_0^2 - \rho g z + \text{const} .$$

Thus at the central point on the surface it is $p_0 = -\rho g z_0 + \text{const}$, hence

$$p - p_0 = \frac{1}{2}\rho r^2 \omega_0^2 - \rho g(z_0 - z) .$$

Now since $p = p_0$ at any point on the surface, the equation for the surface is

$$0 = \frac{1}{2}\rho r^2 \omega_0^2 - \rho g(z_0 - z) \quad \text{or} \quad z = \frac{\omega_0^2}{2g} r^2 + z_0 ,$$

a parabola for any θ — i.e. the surface is a paraboloid of revolution about the vertical axis.

A2) The self-gravitation depends upon the assumed density distribution, which may not be axisymmetric (at least initially). Thus let us again adopt a z -axis of rotation but resolve the equilibrium equation in Cartesian coordinate, so

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = x|\boldsymbol{\omega}_0|^2 - \frac{\partial V}{\partial x} , \quad \frac{1}{\rho} \frac{\partial p}{\partial y} = y|\boldsymbol{\omega}_0|^2 - \frac{\partial V}{\partial y} , \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = -\frac{\partial V}{\partial z} ,$$

where $V(x, y, z)$ such that $\mathbf{g} = -\nabla V$ denotes the gravitational potential. For the homogeneous fluid ($\rho = \text{const}$), integration yields

$$\frac{p}{\rho} = \frac{1}{2}(x^2 + y^2)|\boldsymbol{\omega}_0|^2 - V + \text{const} ,$$

Now the Divergence Theorem (1.60) yields

$$\int_V \nabla^2 p \, d\tau = \int_S d\mathbf{S} \cdot \nabla p = \int_S \frac{\partial p}{\partial n} \, dS ,$$

where $\partial p / \partial n$ denotes the normal outward pressure gradient at the surface S of any region of fluid. We have

$$\begin{aligned} \nabla^2 p &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) p = \rho (2|\boldsymbol{\omega}_0|^2 - \nabla^2 V) \\ &= 2\rho(|\boldsymbol{\omega}_0|^2 - 2\pi G\rho) \end{aligned}$$

using Newton's law of gravitation $\nabla^2 V = 4\pi G\rho$, whence

$$\int_S \frac{\partial p}{\partial n} \, dS = 2\rho(|\boldsymbol{\omega}_0|^2 - 2\pi G\rho) \int_V d\tau .$$

This last result applied to any small internal region implies that the fluid pressure cannot be a maximum at any point if $|\boldsymbol{\omega}_0|^2 > 2\pi G\rho$, and cannot

be a minimum if $|\boldsymbol{\omega}_0|^2 < 2\pi G\rho$. Thus if the pressure is zero at the fluid boundary, it cannot be positive in the first case nor negative in the second. Moreover, when $|\boldsymbol{\omega}_0|^2 = 2\pi\rho$ we have $\nabla^2 p = 0$ and hence $p = 0$ everywhere throughout the fluid, if the pressure is zero at the boundary. Consequently, there is an upper limit $\sqrt{2\pi G\rho}$ to the angular velocity of the fluid, a conclusion reached by the mathematicians Poincaré and Lyapunov in the late nineteenth century.

A3) Crossing terms in (2.17) with $\mathbf{r}$ produces

$$\begin{aligned} \mathbf{r} \times \frac{\partial(\rho\mathbf{v})}{\partial t} &= \frac{\partial}{\partial t}(\mathbf{r} \times \rho\mathbf{v}) \text{ as } \mathbf{r} \text{ is held constant in the} \\ &\quad \text{partial time derivative — cf. (2.23) ;} \\ \mathbf{r} \times \nabla \cdot (\rho\mathbf{v}\mathbf{v}) &= \nabla \cdot (\rho\mathbf{v})\mathbf{r} \times \mathbf{v} - \rho\mathbf{v} \cdot (\nabla\mathbf{v}) \times \mathbf{r}, \quad \text{identical to} \\ \nabla \cdot (\rho\mathbf{v}\mathbf{r} \times \mathbf{v}) &= \nabla \cdot (\rho\mathbf{v})\mathbf{r} \times \mathbf{v} + \rho\mathbf{v} \cdot \nabla(\mathbf{r} \times \mathbf{v}) \quad \text{because} \\ \nabla(\mathbf{r} \times \mathbf{v}) &= (\nabla\mathbf{r}) \times \mathbf{v} - (\nabla\mathbf{v}) \times \mathbf{r} = \mathbf{I} \times \mathbf{v} - (\nabla\mathbf{v}) \times \mathbf{r} \\ \text{and } \mathbf{v} \cdot \mathbf{I} \times \mathbf{v} &= \mathbf{v} \times \mathbf{v} = 0 \text{ leaves only the second term; and} \\ \mathbf{r} \times \nabla \cdot \mathbf{p} &= \nabla \cdot (\mathbf{r} \times \mathbf{p}), \text{ since } (\nabla \times \mathbf{r}) \cdot \mathbf{p} = 0 \text{ because} \\ \nabla \times \mathbf{r} &= 0; \text{ and } \mathbf{r} \times \mathbf{p} = -\mathbf{p} \times \mathbf{r} \text{ since } \mathbf{p} \text{ is symmetric.} \end{aligned}$$

(The equation sought then follows on collecting terms.)

Section 2.5

A1) Let us consider the difference

$$d \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C \mathbf{f}(\mathbf{r}', t + dt) \cdot d\mathbf{r}' - \oint_C \mathbf{f}(\mathbf{r}, t) \cdot d\mathbf{r}$$

during the infinitesimal time interval dt . The fluid particles at each end of the material line element $d\mathbf{r}$ are respectively displaced $\mathbf{v}dt$ and $(\mathbf{v} + d\mathbf{v})dt$, so the material line element $d\mathbf{r}$ changes by $d\mathbf{v}dt$, where $d\mathbf{v} = d\mathbf{r} \cdot \nabla\mathbf{v}$. The line element at time $t + dt$ is therefore $d\mathbf{r}' = d\mathbf{r} + d\mathbf{r} \cdot \nabla\mathbf{v} dt$, and in addition

$$\mathbf{f}(\mathbf{r}', t + dt) = \mathbf{f}(\mathbf{r}, t) + \frac{d\mathbf{f}(\mathbf{r}, t)}{dt} dt$$

such that (retaining terms linear in dt)

$$\begin{aligned} d \oint_C \mathbf{f} \cdot d\mathbf{r} &= \oint_C \left(\frac{d\mathbf{f}}{dt} \cdot d\mathbf{r} + \mathbf{f} \cdot (d\mathbf{r} \cdot \nabla)\mathbf{v} \right) dt \\ &= \oint_C \left(\frac{d\mathbf{f}}{dt} \cdot d\mathbf{r} + d\mathbf{r} \cdot \nabla(\mathbf{f} \cdot \mathbf{v}) - \mathbf{v} \cdot d\mathbf{r} \cdot \nabla\mathbf{f} \right) dt. \end{aligned}$$

The contribution from the second term in the integrand of the last integral vanishes since $\mathbf{f} \cdot \mathbf{v}$ has the same value at the start and end of the closed

path of integration C , whence

$$\frac{d}{dt} \oint_C \mathbf{f} \cdot d\mathbf{r} = \oint_C \left(\frac{d\mathbf{f}}{dt} \cdot d\mathbf{r} - \mathbf{v} \cdot d\mathbf{r} \cdot \nabla \mathbf{f} \right) = \oint_C \left(\frac{d\mathbf{f}}{dt} - (\nabla \mathbf{f}) \cdot \mathbf{v} \right) \cdot d\mathbf{r}.$$

Section 2.6

A1) $n_s \langle \mathbf{w} \rangle = \int f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}) d\mathbf{c} = \int f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{c} d\mathbf{c} - \mathbf{v} \int f_s(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} = n_s \langle \mathbf{c} \rangle - n_s \mathbf{v}$ such that $\mathbf{u}_s = \mathbf{v}_s - \mathbf{v}$ or $\mathbf{v}_s = \mathbf{v} + \mathbf{u}_s$. Thus we have

$$\begin{aligned} \mathbf{P}_s(\mathbf{r}, t) &= \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &= \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{w} - \mathbf{u}_s) (\mathbf{w} - \mathbf{u}_s) d\mathbf{c} \\ &= m_s \left(\int f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} \mathbf{w} d\mathbf{c} - \int f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} d\mathbf{c} \mathbf{u}_s \right. \\ &\quad \left. - \mathbf{u}_s \int f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} d\mathbf{c} + \int f_s(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} \mathbf{u}_s \mathbf{u}_s \right) \\ &= \mathbf{p}_s - m_s (n_s \mathbf{u}_s \mathbf{u}_s + n_s \mathbf{u}_s \mathbf{u}_s - n_s \mathbf{u}_s \mathbf{u}_s) = \mathbf{p}_s - m_s n_s \mathbf{u}_s \mathbf{u}_s. \end{aligned}$$

Setting $\mathbf{v} + \mathbf{u}_s$ for $\mathbf{v}_s$ and $\mathbf{p}_s - \rho_s \mathbf{u}_s \mathbf{u}_s$ for $\mathbf{P}_s$ in the given equation gives

$$\begin{aligned} \rho_s \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial(\rho_s \mathbf{u}_s)}{\partial t} - \mathbf{u}_s \frac{\partial \rho_s}{\partial t} + \rho_s (\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{u}_s + \mathbf{u}_s \cdot \nabla \mathbf{v} + \mathbf{u}_s \cdot \nabla \mathbf{u}_s) \\ + \nabla \cdot \mathbf{p}_s - \mathbf{u}_s \nabla \cdot (\rho_s \mathbf{u}_s) - \rho_s \mathbf{u}_s \cdot \nabla \mathbf{u}_s \\ - n_s e_s \mathbf{E} - n_s e_s \mathbf{v} \times \mathbf{B} - n_s e_s \mathbf{u}_s \times \mathbf{B} - m_s n_s \mathbf{g} - \mathbf{F}_s^{\text{coll}} = 0. \end{aligned}$$

Thus cancelling the $\rho_s \mathbf{u}_s \cdot \nabla \mathbf{u}_s$ terms and invoking (2.46) such that

$$-\mathbf{u}_s \frac{\partial \rho_s}{\partial t} = \mathbf{u}_s \nabla \cdot (\rho_s \mathbf{v}) + \mathbf{u}_s \nabla \cdot (\rho_s \mathbf{u}_s) - \mathbf{u}_s \mathcal{C}_s(m_s),$$

we have

$$\begin{aligned} \rho_s \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + \frac{\partial(\rho_s \mathbf{u}_s)}{\partial t} + \mathbf{u}_s \nabla \cdot (\rho_s \mathbf{v}) + \rho_s \mathbf{v} \cdot \nabla \mathbf{u}_s + \nabla \cdot \mathbf{p}_s \\ + \rho_s \mathbf{u}_s \cdot \nabla \mathbf{v} - n_s e_s (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - n_s e_s \mathbf{u}_s \times \mathbf{B} - m_s n_s \mathbf{g} - \mathbf{F}_s^{\text{coll}} = 0, \end{aligned}$$

which on grouping some terms is (2.47) with $\mathbf{f}_s$ given below (2.43) and $\mathcal{C}_s(m_s \mathbf{w})$ suitably identified.

A2) Writing $d/dt = \partial/\partial t + \mathbf{v}_s \cdot \nabla$ for convenience and adopting $P_s = n_s k T_s$ gives

$$\begin{aligned} \frac{dP_s}{dt} &= \frac{d}{dt} (n_s k T_s) = n_s k \frac{dT_s}{dt} + k T_s \frac{dn_s}{dt} \\ &= n_s k \frac{dT_s}{dt} - k T_s n_s \nabla \cdot \mathbf{v}_s = n_s k \frac{dT_s}{dt} - P_s \nabla \cdot \mathbf{v}_s \end{aligned}$$

and hence the heat balance equation rewritten as

$$\frac{\partial(\frac{3}{2}P_s)}{\partial t} + \nabla \cdot (\frac{3}{2}P_s \mathbf{v}_s + \mathbf{Q}_s) + \mathbf{P}_s : \nabla \mathbf{v}_s = G_s^{\text{coll}}.$$

Now we have the thermal flux vector

$$\begin{aligned} \mathbf{Q}_s &= \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &= \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{w} - \mathbf{u}_s) \cdot (\mathbf{w} - \mathbf{u}_s) (\mathbf{w} - \mathbf{u}_s) d\mathbf{c} \\ &= \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (w^2 - 2\mathbf{u}_s \cdot \mathbf{w} + u_s^2) (\mathbf{w} - \mathbf{u}_s) d\mathbf{c} \\ &= \frac{1}{2} \left(\int m_s f_s(\mathbf{r}, \mathbf{c}, t) w^2 \mathbf{w} d\mathbf{c} - 2\mathbf{u}_s \cdot \int m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} \mathbf{w} d\mathbf{c} \right. \\ &\quad \left. + u_s^2 \int m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} d\mathbf{c} - \mathbf{u}_s \int m_s f_s(\mathbf{r}, \mathbf{c}, t) w^2 d\mathbf{c} \right. \\ &\quad \left. + 2\mathbf{u}_s \mathbf{u}_s \cdot \int m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{w} d\mathbf{c} - u_s^2 \mathbf{u}_s \int m_s f_s(\mathbf{r}, \mathbf{c}, t) d\mathbf{c} \right) \\ &= \mathbf{q}_s - \mathbf{p}_s \cdot \mathbf{u}_s - \frac{3}{2} p_s \mathbf{u}_s + \rho_s u_s^2 \mathbf{u}_s; \end{aligned}$$

and recalling also that $\mathbf{v}_s = \mathbf{v} + \mathbf{u}_s$ and $\mathbf{P}_s = \mathbf{p}_s - \rho_s \mathbf{u}_s \mathbf{u}_s$ such that in addition $P_s = p_s - (1/3)\rho_s u_s^2$, we rewrite the above heat balance equation as

$$\begin{aligned} &\frac{\partial(\frac{3}{2}p_s)}{\partial t} + \nabla \cdot \left(\frac{3}{2}p_s \mathbf{v} + \mathbf{q}_s \right) + \mathbf{p}_s : \nabla \mathbf{v} - \frac{\partial(\frac{1}{2}\rho_s u_s^2)}{\partial t} - \nabla \cdot \left(\frac{1}{2}\rho_s u_s^2 \mathbf{v} \right) \\ &+ \nabla \cdot \left(\frac{3}{2}p_s \mathbf{u}_s \right) - \nabla \cdot \left(\frac{1}{2}\rho_s u_s^2 \mathbf{u}_s \right) - \nabla \cdot (\mathbf{p}_s \cdot \mathbf{u}_s) - \nabla \cdot \left(\frac{3}{2}p_s \mathbf{u}_s \right) \\ &+ \nabla \cdot (\rho_s u_s^2 \mathbf{u}_s) + \mathbf{p}_s : \nabla \mathbf{u}_s - \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{v} - \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{u}_s = G_s^{\text{coll}} \end{aligned}$$

or

$$\begin{aligned} &\frac{\partial(\frac{3}{2}p_s)}{\partial t} + \nabla \cdot \left(\frac{3}{2}p_s \mathbf{v} + \mathbf{q}_s \right) + \mathbf{p}_s : \nabla \mathbf{v} - \frac{\partial(\frac{1}{2}\rho_s u_s^2)}{\partial t} - \nabla \cdot \left(\frac{1}{2}\rho_s u_s^2 \mathbf{v} \right) \\ &- \nabla \cdot \left(\frac{1}{2}\rho_s u_s^2 \mathbf{u}_s \right) - \nabla \cdot (\mathbf{p}_s \cdot \mathbf{u}_s) + \nabla \cdot (\rho_s u_s^2 \mathbf{u}_s) \\ &+ \mathbf{p}_s : \nabla \mathbf{u}_s - \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{v} - \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{u}_s = G_s^{\text{coll}} \end{aligned}$$

after the obvious cancellation. Taking the dot product of the momentum equation (2.47) with $\mathbf{u}_s$, we obtain the energy transport equation

$$\begin{aligned} &\frac{\partial(\frac{1}{2}\rho_s u_s^2)}{\partial t} + \frac{1}{2} u_s^2 \frac{\partial \rho_s}{\partial t} + \nabla \cdot \left(\frac{1}{2}\rho_s u_s^2 \mathbf{v} \right) + \frac{1}{2} u_s^2 \nabla \cdot (\rho_s \mathbf{v}) + \nabla \cdot (\mathbf{p}_s \cdot \mathbf{u}_s) \\ &- \mathbf{p}_s : \nabla \mathbf{u}_s - \rho_s \mathbf{f}_s \cdot \mathbf{u}_s + \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{v} = \mathcal{C}_s(m_s \mathbf{w}) \cdot \mathbf{u}_s, \end{aligned}$$

in which we may combine two terms into one — i.e.

$$\frac{1}{2}u_s^2 \frac{\partial \rho_s}{\partial t} + \frac{1}{2}u_s^2 \nabla \cdot (\rho_s \mathbf{v}) = \frac{1}{2}u_s^2 [\mathcal{C}_s(m_s) - \nabla \cdot (\rho_s \mathbf{u}_s)] ,$$

on invoking (2.46). Then adding this energy transport equation to the above heat balance equation, we obtain (2.48) on suitably identifying the collision term G_s^{coll} and noting that

$$\begin{aligned} \nabla \cdot \left(\frac{1}{2} \rho_s u_s^2 \mathbf{u}_s \right) &= \frac{1}{2} u_s^2 \nabla \cdot (\rho_s \mathbf{u}_s) + \rho_s \mathbf{u}_s \cdot \nabla \left(\frac{1}{2} u_s^2 \right) \\ \text{and } \rho_s \mathbf{u}_s \mathbf{u}_s : \nabla \mathbf{u}_s &= \rho_s \mathbf{u}_s \cdot \nabla \left(\frac{1}{2} u_s^2 \right) . \end{aligned}$$

A3) We have

$$\begin{aligned} \nabla \cdot \mathbf{Q}_s &= - (\nabla \cdot \mathbf{v}_s) \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &\quad + \int m_s (\mathbf{c} - \mathbf{v}_s) \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= - \mathbf{P}_s \nabla \cdot \mathbf{v}_s - \mathbf{v}_s \cdot \nabla \mathbf{P}_s \\ &\quad + \int m_s \mathbf{c} \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} , \end{aligned}$$

in which $-\mathbf{P}_s \nabla \cdot \mathbf{v}_s - \mathbf{v}_s \cdot \nabla \mathbf{P}_s = -\nabla \cdot (\mathbf{v}_s \mathbf{P}_s)$ and

$$\begin{aligned} &\int m_s \mathbf{c} \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= -\mathbf{v}_s \overset{\leftarrow}{\nabla} \cdot \left(\int m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{c} (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \right) \\ &\quad + \int m_s (\mathbf{c} - \mathbf{v}_s) \mathbf{c} \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= -\mathbf{v}_s \overset{\leftarrow}{\nabla} \cdot \left(\int m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{c} (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \right) \\ &\quad - \left(\int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \mathbf{c} d\mathbf{c} \right) \cdot \nabla \mathbf{v}_s \\ &\quad + \int m_s (\mathbf{c} \cdot \nabla f_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} , \\ &= -\mathbf{v}_s \overset{\leftarrow}{\nabla} \cdot \mathbf{P}_s - \mathbf{P}_s \cdot \nabla \mathbf{v}_s + \int m_s (\mathbf{c} \cdot \nabla f_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} , \end{aligned}$$

where the last term on the right-hand side can be evaluated by invoking the Boltzmann-type equation to replace $\mathbf{c} \cdot \nabla f_s$.

Thus we obtain

$$\begin{aligned}
& \int m_s (\mathbf{c} \cdot \nabla f_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\
&= \int m_s \left(\frac{\delta f_s}{\delta t} - \frac{\partial f_s}{\partial t} - \mathbf{a}_s(\mathbf{r}, \mathbf{c}, t) \cdot \nabla_{\mathbf{c}} f_s \right) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\
&= \mathbf{G}_s^{\text{coll}} - \frac{\partial \mathbf{P}_s}{\partial t} - \frac{e_s}{m_s} \int \mathbf{c} \times \mathbf{B} \cdot (\nabla_{\mathbf{c}} f_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\
&= \mathbf{G}_s^{\text{coll}} - \frac{\partial \mathbf{P}_s}{\partial t} + 2 \frac{e_s}{m_s} \{ \mathbf{P}_s \times \mathbf{B} \}
\end{aligned}$$

on observing that $\mathbf{c} \times \mathbf{B} \cdot (\nabla_{\mathbf{c}} f_s) = \mathbf{B} \times (\nabla_{\mathbf{c}} f_s) \cdot \mathbf{c} = -\mathbf{c} \cdot (\nabla_{\mathbf{c}} f_s) \times \mathbf{B}$ and then integrating by parts (noting the gravity and electric field contributions from $\mathbf{a}_s$ vanish), whence the result sought on rearranging the equation

$$\begin{aligned}
\nabla \cdot \mathbf{Q}_s &= \mathbf{G}_s^{\text{coll}} - \frac{\partial \mathbf{P}_s}{\partial t} - \nabla \cdot (\mathbf{v}_s \mathbf{P}_s) \\
&\quad - \mathbf{v}_s \overset{\leftarrow}{\nabla} \cdot \mathbf{P}_s - \mathbf{P}_s \cdot \nabla \mathbf{v}_s + 2 \frac{e_s}{m_s} \{ \mathbf{P}_s \times \mathbf{B} \}
\end{aligned}$$

that follows on assembling the above terms.

Notes:

1. In the case of a plasma consisting of electrons and only one ion species, to be discussed further in Sect.2.10, the alternative reference velocities for the ions are almost identical (corresponding to $\mathbf{u}_i \simeq 0$). Thus $\mathbf{P}_i \simeq \mathbf{p}_i$, and the traceless part of this result (in terms of $\{ \mathbf{P}_i \}$) is identical to equation (2.49) in $\mathbf{t}_i$. However, the alternative reference velocities for the electron component are quite distinct, corresponding to inter-species — i.e. $\mathbf{u}_e \neq 0$ and $\mathbf{P}_e = \mathbf{p}_e - \rho_e \mathbf{u}_e \mathbf{u}_e$.
2. $\mathbf{Q}_s$ is analogous to $\mathbf{H}_s$, and its traceless part is analogous to $\mathbf{h}_s = \{ \mathbf{H}_s \}$ in (2.49). Further, a contraction of the next higher order equation involving the fourth order tensor

$$\mathbf{R}_s = \int m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}$$

produces the alternative equation corresponding to (2.50), where $\mathbf{f}_s$ is eliminated using (2.47) and also (2.46) when $\mathbf{u}_s \neq 0$. Thus the equation for the thermal flux vector $\mathbf{Q}_s$ defined in (Q2) is

$$\begin{aligned}
& \frac{\partial \mathbf{Q}_s}{\partial t} + \nabla \cdot (\mathbf{v}_s \mathbf{Q}_s + \tilde{\mathbf{R}}_s) + \mathbf{Q}_s : \nabla \mathbf{v}_s + \mathbf{Q}_s \cdot \nabla \mathbf{v}_s - \frac{e_s}{m_s} \mathbf{Q}_s \times \mathbf{B} \\
& - \frac{3}{2} \frac{P_s}{\rho_s} \nabla \cdot \mathbf{P}_s - \frac{1}{\rho_s} (\nabla \cdot \mathbf{P}_s) \cdot \mathbf{P}_s = \mathbf{H}_s^{\text{coll}} - \frac{3}{2} \frac{P_s}{\rho_s} \mathbf{F}_s^{\text{coll}} - \mathbf{F}_s^{\text{coll}} \cdot \mathbf{P}_s
\end{aligned}$$

involving the analogue of ℓ_s , the second order tensor contraction of $\mathbf{R}_s$:

$$\tilde{\mathbf{R}}_s = \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}.$$

This last equation in $\mathbf{Q}_s$ may be derived in a rather similar fashion. Thus

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{R}}_s &= \nabla \cdot \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &= -(\nabla \cdot \mathbf{v}_s) \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &\quad + \int \frac{1}{2} m_s (\mathbf{c} - \mathbf{v}_s) \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= -\nabla \cdot (\mathbf{v}_s \mathbf{Q}_s) \\ &\quad + \int \frac{1}{2} m_s \mathbf{c} \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c}, \end{aligned}$$

since $-\mathbf{Q}_s \nabla \cdot \mathbf{v}_s - \mathbf{v}_s \cdot \nabla \mathbf{Q}_s = -\nabla \cdot (\mathbf{v}_s \mathbf{Q}_s)$ and

$$\begin{aligned} &\int \frac{1}{2} m_s \mathbf{c} \cdot \nabla [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= -\mathbf{v}_s \overset{\leftarrow}{\nabla} : \int \frac{1}{2} m_s f_s(\mathbf{r}, \mathbf{c}, t) \mathbf{c} (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &\quad + \int \frac{1}{2} m_s (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} \cdot \nabla) [f_s(\mathbf{r}, \mathbf{c}, t) (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &= -\mathbf{Q}_s : \nabla \mathbf{v}_s - \mathbf{Q}_s \cdot \nabla \mathbf{v}_s - \mathbf{P}_s \cdot (\mathbf{v}_s \cdot \nabla \mathbf{v}_s) - \frac{3}{2} P_s \mathbf{v}_s \cdot \nabla \mathbf{v}_s \\ &\quad + \int \frac{1}{2} m_s (\mathbf{c} \cdot \nabla f_s) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}, \end{aligned}$$

where once again the last term can be evaluated by invoking the Boltzmann-type equation — i.e. this last integral is

$$\int \frac{m_s}{2} \left(\frac{\delta f_s}{\delta t} - \frac{\partial f_s}{\partial t} - \mathbf{a}_s(\mathbf{r}, \mathbf{c}, t) \cdot \nabla_{\mathbf{c}} f_s \right) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c}$$

yielding

$$\begin{aligned} &\mathbf{H}_s^{\text{coll}} - \frac{\partial \mathbf{Q}_s}{\partial t} + \int \frac{1}{2} m_s f_s \frac{\partial}{\partial t} [(\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s)] d\mathbf{c} \\ &- \int \frac{m_s}{2} \left[\mathbf{g} + \frac{e_s}{m_s} (\mathbf{E} + \mathbf{c} \times \mathbf{B}) \right] \cdot (\nabla_{\mathbf{c}} f_s) (\mathbf{c} - \mathbf{v}_s) \cdot (\mathbf{c} - \mathbf{v}_s) (\mathbf{c} - \mathbf{v}_s) d\mathbf{c} \\ &= \mathbf{H}_s^{\text{coll}} - \frac{\partial \mathbf{Q}_s}{\partial t} + \frac{e_s}{m_s} \mathbf{Q}_s \times \mathbf{B} - \mathbf{P}_s \cdot \frac{\partial \mathbf{v}_s}{\partial t} - \frac{3}{2} P_s \frac{\partial \mathbf{v}_s}{\partial t} \\ &+ \mathbf{P}_s \cdot \left[\mathbf{g} + \frac{e_s}{m_s} (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \right] + \frac{3}{2} P_s \left[\mathbf{g} + \frac{e_s}{m_s} (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) \right]. \end{aligned}$$

On assembling the above terms and invoking the equation of motion in $\mathbf{v}_s$ derived in (A1) above, the equation in $\mathbf{Q}_s$ analogous to (2.50) follows.

Section 2.9

A1) If $\mathbf{t}$ is a dyadic and $\mathbf{a}$ is a vector, the cross product $\mathbf{t} \times \mathbf{a} = t_{ij}a_k \epsilon_{jkl} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_l$, where ϵ_{jkl} denotes the Levi-Civita symbol and $\{\hat{\mathbf{e}}_i\}$ any Cartesian (self-reciprocal) basis set (in the usual index notation and convention that repeated indices denotes summation over 1, 2, 3) as discussed in Sect.1.2.. If $\mathbf{t}$ is symmetric ($\mathbf{t}^T = \mathbf{t}$), then $(\mathbf{t} \times \mathbf{a})^T = -\mathbf{a} \times \mathbf{t}$ and $\mathbf{l} : \mathbf{t} \times \mathbf{a} = 0$, where $\mathbf{l} = \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i$ is the unit dyadic; so on applying the bracket operator defined in the footnote on page 41, we immediately have $2\{\mathbf{t} \times \mathbf{a}\} = \mathbf{t} \times \mathbf{a} - \mathbf{a} \times \mathbf{t}$ and thus

$$4\{\{\mathbf{t} \times \mathbf{a}\} \times \mathbf{a}\} = (\mathbf{t} \times \mathbf{a}) \times \mathbf{a} - 2\mathbf{a} \times \mathbf{t} \times \mathbf{a} + \mathbf{a} \times (\mathbf{a} \times \mathbf{t}),$$

on substituting $\{\mathbf{t} \times \mathbf{a}\}$ for $\mathbf{t}$ and noting that brackets are unnecessary in the second triple product on the right-hand side — i.e. $(\mathbf{a} \times \mathbf{t}) \times \mathbf{a} = \mathbf{a} \times (\mathbf{t} \times \mathbf{a})$. Also $2\{\mathbf{t} \times \mathbf{a}\} : \mathbf{a}\mathbf{a} = (\mathbf{t} \times \mathbf{a} - \mathbf{a} \times \mathbf{t}) \cdot \mathbf{a} \cdot \mathbf{a} = 0$.

Now $(\mathbf{t} \times \mathbf{a}) \times \mathbf{a} = \mathbf{t} \cdot \mathbf{a}\mathbf{a} - |\mathbf{a}|^2 \mathbf{t}$ and $\mathbf{a} \times (\mathbf{a} \times \mathbf{t}) = \mathbf{a}\mathbf{a} \cdot \mathbf{t} - |\mathbf{a}|^2 \mathbf{t}$; and if the symmetric dyadic $\mathbf{t}$ is also traceless ($t_{ii} = 0$), on writing $\mathbf{a} \times \mathbf{t} \times \mathbf{a} = a_i t_{jk} a_m \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j \hat{\mathbf{e}}_k \times \hat{\mathbf{e}}_m$ and grouping terms we obtain

$$\mathbf{a} \times \mathbf{t} \times \mathbf{a} = -(\mathbf{t} \cdot \mathbf{a}\mathbf{a} + \mathbf{a}\mathbf{a} \cdot \mathbf{t}) + (\mathbf{l} \mathbf{t} : \mathbf{a}\mathbf{a} + |\mathbf{a}|^2 \mathbf{t}),$$

on noting that $\mathbf{a}\mathbf{a} : \mathbf{l} = |\mathbf{a}|^2$. The previously displayed equation therefore yields the required first identity

$$2\{\{\mathbf{t} \times \mathbf{a}\} \times \mathbf{a}\} = 3\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} - 2|\mathbf{a}|^2 \mathbf{t},$$

since $(\mathbf{t} \cdot \mathbf{a}\mathbf{a})^T = \mathbf{a}\mathbf{a} \cdot \mathbf{t}$ and $(\mathbf{t} \cdot \mathbf{a}\mathbf{a}) : \mathbf{l} = \mathbf{t} : \mathbf{a}\mathbf{a}$.

The result $6\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} \cdot \mathbf{a} = 3\mathbf{t} \cdot \mathbf{a} |\mathbf{a}|^2 + \mathbf{a}\mathbf{t} : \mathbf{a}\mathbf{a}$ then follows on noting that $\mathbf{a} \cdot \mathbf{t} \cdot \mathbf{a} = \mathbf{t} : \mathbf{a}\mathbf{a}$.

Finally, on replacing $\mathbf{t}$ with $\{\mathbf{t} \times \mathbf{a}\}$ in the first identity, and comparing the result with a post cross product of the first identity with the vector $\mathbf{a}$ followed by the bracket operator, we obtain the third identity

$$\{\{\mathbf{t} \times \mathbf{a}\} \cdot \mathbf{a}\mathbf{a}\} = \{\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} \times \mathbf{a}\}.$$

A2) A post cross product of (2.67) with the vector $\mathbf{a}$ followed by the bracket operator yields

$$-4\mu\{\mathbf{s} \times \mathbf{a}\} + 4\{\{\mathbf{t} \times \mathbf{a}\} \times \mathbf{a}\} = 2\{\mathbf{t} \times \mathbf{a}\} = 2\mu\mathbf{s} + \mathbf{t} \quad (*),$$

and the first identity in (Q1) then produces

$$-4\mu\{\mathbf{s} \times \mathbf{a}\} + 6\{\mathbf{t} \cdot \mathbf{a}\mathbf{a}\} - 2\mu\mathbf{s} = (1 + 4|\mathbf{a}|^2)\mathbf{t}.$$

Taking a post dot product of $(*)$ with the vector $\mathbf{a}$, and invoking the result $6\{\mathbf{t} \cdot \mathbf{aa}\} \cdot \mathbf{a} = 3\mathbf{t} \cdot \mathbf{a} |\mathbf{a}|^2 + \mathbf{at} : \mathbf{aa}$ of (Q1), yields

$$(1 + |\mathbf{a}|^2)\mathbf{t} \cdot \mathbf{a} = -4\mu\{\mathbf{s} \times \mathbf{a}\} \cdot \mathbf{a} - 2\mu(\mathbf{s} : \mathbf{aa} \mathbf{a} + \mathbf{s} \cdot \mathbf{a}) ,$$

since $\mathbf{t} : \mathbf{aa} = -2\mu\mathbf{s} : \mathbf{aa}$ follows from (2.67). Finally, eliminating $\mathbf{t} \cdot \mathbf{a}$ between the last two displayed equations and invoking the third identity yields (2.69).

A3) Since $\mathbf{s}$ is symmetric, $(\mathbf{s} \times \hat{\mathbf{b}})^T = -\hat{\mathbf{b}} \times \mathbf{s}$ and $(\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}})^T = (\hat{\mathbf{b}} \cdot \mathbf{s} \hat{\mathbf{b}})^T = \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s}$. Also $\mathbf{s} \times \hat{\mathbf{b}} : \mathbf{I} = 0$ and $\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} : \mathbf{I} = \mathbf{s} : \hat{\mathbf{b}} \hat{\mathbf{b}} = \hat{\mathbf{b}} \cdot \mathbf{s} \cdot \hat{\mathbf{b}}$, so $\{\mathbf{s} \times \hat{\mathbf{b}}\} = \frac{1}{2}(\mathbf{s} \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \mathbf{s})$; and $\{\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}\} \times \hat{\mathbf{b}} = [\frac{1}{2}(\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} + \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s}) - \frac{1}{3}\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} : \mathbf{II}] \times \hat{\mathbf{b}} = \frac{1}{2}\hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}} - \frac{1}{3}\mathbf{s} : \hat{\mathbf{b}} \hat{\mathbf{b}} \mathbf{I} \times \hat{\mathbf{b}}$ gives $\{6\{\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}\} \times \hat{\mathbf{b}}\} = \frac{3}{2}(\hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}})$, as $(\hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}})^T = -\hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}$ and $\mathbf{I} \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \mathbf{I} = 0$. Consequently, the gyroviscous (cross) component

$$\begin{aligned} \mathbf{t}_g &= \frac{\mu}{2|\mathbf{a}|} [\hat{\mathbf{b}} \times \mathbf{s} - \mathbf{s} \times \hat{\mathbf{b}} + 3(\hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} - \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}})] \\ &= \frac{\mu}{2|\mathbf{a}|} [\hat{\mathbf{b}} \times \mathbf{s} \cdot \mathbf{I}_\perp - \mathbf{I}_\perp \cdot \mathbf{s} \times \hat{\mathbf{b}} + 4(\hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} - \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}})] . \end{aligned}$$

The alternative form for the perpendicular component follows readily from a straightforward evaluation of $\mathbf{s} + 6\{\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}\} - \frac{15}{2}\mathbf{s} : \hat{\mathbf{b}} \hat{\mathbf{b}} (\hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{3}\mathbf{I})$, on noting that $6\{\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}}\} = 3(\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} + \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s}) - 2\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} : \mathbf{II}$ and $\mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} : \mathbf{I} = \mathbf{s} : \hat{\mathbf{b}} \hat{\mathbf{b}} = \hat{\mathbf{b}} \cdot \mathbf{s} \cdot \hat{\mathbf{b}}$.

A4) The following is a derivation from the original forms (2.71)–(2.73).

Since $\mathbf{s} : \hat{\mathbf{b}} \hat{\mathbf{b}} = \mathbf{s} : \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 = s_{33} = -(s_{11} + s_{22})$, in matrix form the representation of the parallel component

$$\mathbf{t}_\parallel = -3\mu\mathbf{s} : \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \{\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} = -3\mu s_{33} \left(\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 - \frac{1}{3}\mathbf{I} \right) = \mu s_{33} (\mathbf{I}_\perp - 2\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3),$$

may be written

$$-2\mu \begin{bmatrix} \frac{1}{2}(s_{11} + s_{22}) & 0 & 0 \\ 0 & \frac{1}{2}(s_{11} + s_{22}) & 0 \\ 0 & 0 & s_{33} \end{bmatrix} .$$

The gyroviscous (cross) component

$$\mathbf{t}_g = -\frac{\mu}{|\mathbf{a}|} \{\mathbf{s} \times \hat{\mathbf{e}}_3 + 6\{\mathbf{s} \cdot \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} \times \hat{\mathbf{e}}_3\}$$

involves the traceless part of $\mathbf{s} \times \hat{\mathbf{e}}_3 = s_{12}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 - s_{11}\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 + s_{22}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_1 - s_{12}\hat{\mathbf{e}}_2\hat{\mathbf{e}}_2 + s_{23}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 - s_{13}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2$ plus $6\{\mathbf{s} \cdot \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} \times \hat{\mathbf{e}}_3 = 2s_{33}(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2 - \hat{\mathbf{e}}_2\hat{\mathbf{e}}_1) + 3s_{23}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1 - 3s_{13}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2$, on noting that

$$\begin{aligned} 6\{\mathbf{s} \cdot \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} &= 3s_{13}(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_1) \\ &\quad + 3s_{23}(\hat{\mathbf{e}}_2\hat{\mathbf{e}}_3 + \hat{\mathbf{e}}_3\hat{\mathbf{e}}_2) - 2s_{33}(\hat{\mathbf{e}}_1\hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2\hat{\mathbf{e}}_2) + 4s_{33}\hat{\mathbf{e}}_3\hat{\mathbf{e}}_3 . \end{aligned}$$

Thus in matrix form the representation of $\mathbf{t}_g$ may be written

$$-\frac{\mu}{|\mathbf{a}|} \begin{bmatrix} s_{12} & -\frac{1}{2}(s_{11} - s_{22}) & 2s_{23} \\ -\frac{1}{2}(s_{11} - s_{22}) & -s_{12} & -2s_{13} \\ 2s_{23} & -2s_{13} & 0 \end{bmatrix}.$$

Finally, the representation for the perpendicular component

$$\mathbf{t}_\perp = \frac{\mu}{2|\mathbf{a}|^2} \left(\mathbf{s} + 6\{\mathbf{s} \cdot \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} - \frac{15}{2} \mathbf{s} : \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \{\hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3\} \right)$$

may be written in the matrix form

$$-\frac{\mu}{2|\mathbf{a}|^2} \begin{bmatrix} \frac{1}{2}(s_{11} - s_{22}) & s_{12} & 4s_{13} \\ s_{12} & -\frac{1}{2}(s_{11} - s_{22}) & 4s_{23} \\ 4s_{13} & 4s_{23} & 0 \end{bmatrix}.$$

If the set of three viscosity coefficients is taken to be

$$\{ \mu_\parallel = 2\mu, \quad \mu_g = -\mu/|\mathbf{a}|, \quad \mu_\perp = \mu/(2|\mathbf{a}|^2) \},$$

the combined result may be written out as

$$\begin{aligned} t_{11} &= -\frac{1}{2}\mu_\parallel(s_{11} + s_{22}) + \mu_g s_{12} - \frac{1}{2}\mu_\perp(s_{11} - s_{22}) \\ t_{12} &= t_{21} = -\frac{1}{2}\mu_g(s_{11} - s_{22}) - \mu_\perp s_{12} \\ t_{13} &= t_{31} = 2\mu_g s_{23} - 4\mu_\perp s_{13} \\ t_{22} &= -\frac{1}{2}\mu_\parallel(s_{11} + s_{22}) - \mu_g s_{12} + \frac{1}{2}\mu_\perp(s_{11} - s_{22}) \\ t_{23} &= t_{32} = -2\mu_g s_{13} - 4\mu_\perp s_{23} \\ t_{33} &= -\mu_\parallel s_{33}. \end{aligned}$$

These results also follow from the alternative forms (2.76)–(2.78) of course.

Section 2.10

A1) With $\mathbf{t} = \mathbf{t}_\parallel = (3/2)t_{\parallel,\parallel}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}$ where $t_{\parallel,\parallel} = \mathbf{t}_\parallel : \hat{\mathbf{b}}\hat{\mathbf{b}}$, we have

$$\frac{d\mathbf{t}}{dt} = \frac{3}{2} \left(\frac{dt_{\parallel,\parallel}}{dt} \right) \{\hat{\mathbf{b}}\hat{\mathbf{b}}\} + \frac{3}{2} t_{\parallel,\parallel} \frac{d}{dt}(\hat{\mathbf{b}}\hat{\mathbf{b}}) \Rightarrow \left(\frac{d\mathbf{t}}{dt} \right) : \hat{\mathbf{b}}\hat{\mathbf{b}} = \frac{dt_{\parallel,\parallel}}{dt},$$

$$\text{since} \quad \left(\frac{d}{dt}(\hat{\mathbf{b}}\hat{\mathbf{b}}) \right) : \hat{\mathbf{b}}\hat{\mathbf{b}} = 2\hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{b}}}{dt} = \frac{d}{dt}(\hat{\mathbf{b}} \cdot \hat{\mathbf{b}}) = 0.$$

We also have $2\{\mathbf{t} \times \hat{\mathbf{b}}\} = \mathbf{t} \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \mathbf{t}$ since $\mathbf{t} \times \hat{\mathbf{b}} : \mathbf{I} = 0$ ($\mathbf{t}$ symmetric) such that $\{\mathbf{t} \times \hat{\mathbf{b}}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} = 0$, so the cross term does not remain as one would expect.

Now let us proceed to evaluate $2\{\mathbf{p} \cdot \nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} = (2p\{\nabla \mathbf{v}\} + 2\{\mathbf{t} \cdot \nabla \mathbf{v}\}) : \hat{\mathbf{b}}\hat{\mathbf{b}}$.

We have $2p\{\nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} = 2p\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} - (2/3)p\nabla \cdot \mathbf{v}$ for the first term; and for the second term, from $\mathbf{t} = \mathbf{t}_{\parallel} = (3/2)t_{\parallel,\parallel}\{\hat{\mathbf{b}}\hat{\mathbf{b}}\}$ that

$$\begin{aligned}\mathbf{t} \cdot \nabla \mathbf{v} &= \frac{3}{2}t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla \mathbf{v} - \frac{1}{2}t_{\parallel,\parallel}\nabla \mathbf{v}, \\ (\nabla \mathbf{v})^T \cdot \mathbf{t} &= \frac{3}{2}t_{\parallel,\parallel}(\nabla \mathbf{v})^T \cdot \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{2}t_{\parallel,\parallel}(\nabla \mathbf{v})^T;\end{aligned}$$

and from $\mathbf{t} \cdot (\nabla \mathbf{v}) : \mathbf{I} = \mathbf{t} : \nabla \mathbf{v}$ or directly that

$$-\frac{2}{3}\mathbf{t} \cdot (\nabla \mathbf{v}) : \mathbf{I} = -t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + \frac{1}{3}t_{\parallel,\parallel}\nabla \cdot \mathbf{v}.$$

Thus noting also that

$$\begin{aligned}(\mathbf{t} \cdot \nabla \mathbf{v}) : \hat{\mathbf{b}}\hat{\mathbf{b}} &= \frac{3}{2}t_{\parallel,\parallel}(\hat{\mathbf{b}} \cdot \nabla \mathbf{v}) \cdot \hat{\mathbf{b}} - \frac{1}{2}t_{\parallel,\parallel}(\nabla \mathbf{v}) : \hat{\mathbf{b}}\hat{\mathbf{b}} \\ &= t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} = ((\nabla \mathbf{v})^T \cdot \mathbf{t}) : \hat{\mathbf{b}}\hat{\mathbf{b}},\end{aligned}$$

we obtain

$$\begin{aligned}2\{\mathbf{t} \cdot \nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} &= (\mathbf{t} \cdot \nabla \mathbf{v}) : \hat{\mathbf{b}}\hat{\mathbf{b}} + ((\nabla \mathbf{v})^T \cdot \mathbf{t}) : \hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{2}{3}(\mathbf{t} \cdot \nabla \mathbf{v}) : \mathbf{I} : \hat{\mathbf{b}}\hat{\mathbf{b}} \\ &= t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} - t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + \frac{1}{3}t_{\parallel,\parallel}\nabla \cdot \mathbf{v} \\ &= t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + \frac{1}{3}t_{\parallel,\parallel}\nabla \cdot \mathbf{v}\end{aligned}$$

and consequently equation (2.79).

A2) Omitting the $\nabla \cdot \mathbf{h}$ term, recalling that $\mathbf{u} \simeq 0$ for the ions and collisional coupling is negligible in an ion-electron plasma, and invoking the result

$$2\{\mathbf{t} \cdot \nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} = t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v} + \frac{1}{3}t_{\parallel,\parallel}\nabla \cdot \mathbf{v}$$

obtained in (A1) above, we have the $\hat{\mathbf{b}}\hat{\mathbf{b}}$ projection of (2.68) as

$$2p\mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}} = 2p\{\nabla \mathbf{v}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} + \frac{4}{3}t_{\parallel,\parallel}\nabla \cdot \mathbf{v} + t_{\parallel,\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} : \nabla \mathbf{v}.$$

On noting once again that $t_{\parallel,\parallel} = \mathbf{t} : \hat{\mathbf{b}}\hat{\mathbf{b}} = -2\mu\mathbf{s} : \hat{\mathbf{b}}\hat{\mathbf{b}}$ is the $\hat{\mathbf{b}}\hat{\mathbf{b}}$ projection of (2.67), the stated result consistent with (2.79) immediately emerges.

A3) The additional electromagnetic contribution involves

$$\begin{aligned}2\{\mathbf{E} \mathbf{j}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} &= [\mathbf{E} \mathbf{j} + \mathbf{j} \mathbf{E} - (2/3)\mathbf{E} \mathbf{j} \cdot \mathbf{I} \mathbf{I}] : \hat{\mathbf{b}}\hat{\mathbf{b}} \\ &= 2\mathbf{E} \cdot \hat{\mathbf{b}} \mathbf{j} \cdot \hat{\mathbf{b}} - (2/3)\mathbf{E} \cdot \mathbf{j}, \text{ and} \\ 2\{\mathbf{v}_e \times \mathbf{B} \mathbf{j}\} : \hat{\mathbf{b}}\hat{\mathbf{b}} &= -(2/3)\mathbf{v}_e \times \mathbf{B} \mathbf{j} \cdot \mathbf{I} \mathbf{I} : \hat{\mathbf{b}}\hat{\mathbf{b}} \\ &= -(2/3)(\mathbf{v}_e \times \mathbf{B}) \cdot \mathbf{j},\end{aligned}$$

hence the further terms included in (2.83) for the electrons.

A4) The parallel component produces the dominant ion viscosity contribution

$$\begin{aligned}\nabla \cdot \mathbf{t}_{\parallel} &= \nabla \cdot \left[\frac{3}{2} t_{\parallel, \parallel} \left(\hat{\mathbf{b}} \hat{\mathbf{b}} - \frac{1}{3} \mathbf{I} \right) \right] \\ &= \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla \left(\frac{3}{2} t_{\parallel, \parallel} \right) - \nabla \left(\frac{1}{2} t_{\parallel, \parallel} \right) + \left(\frac{3}{2} t_{\parallel, \parallel} \right) \nabla \cdot (\hat{\mathbf{b}} \hat{\mathbf{b}})\end{aligned}$$

in the equation of motion (2.56). Then referring to the derivation of the alternative form (2.77) for the lower order gyroviscous (cross) component in (A3) of Sect.2.9, we find

$$\begin{aligned}\mathbf{t}_g &= \frac{\mu}{2|\mathbf{a}|} [\hat{\mathbf{b}} \times \mathbf{s} - \mathbf{s} \times \hat{\mathbf{b}} + 3(\hat{\mathbf{b}} \times \mathbf{s} \cdot \hat{\mathbf{b}} \hat{\mathbf{b}} - \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \mathbf{s} \times \hat{\mathbf{b}})] \\ &= \frac{\mu}{2|\mathbf{a}|} \left[\hat{\mathbf{b}} \times \mathbf{s} \cdot (\mathbf{I} + 3\hat{\mathbf{b}} \hat{\mathbf{b}}) - (\mathbf{I} + 3\hat{\mathbf{b}} \hat{\mathbf{b}}) \cdot \mathbf{s} \times \hat{\mathbf{b}} \right].\end{aligned}$$

Following the hint, on using index notation we have

$$\begin{aligned}\hat{\mathbf{b}} \times [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \cdot (\mathbf{I} + 3\hat{\mathbf{b}} \hat{\mathbf{b}}) \Big|_{ij} &= \epsilon_{ikl} \hat{b}_k (\partial v_m / \partial x_l + \partial v_l / \partial x_m) (\delta_{mj} + 3b_m b_j) \\ &= \epsilon_{ikl} \hat{b}_k \left[\left(2 \frac{\partial v_j}{\partial x_l} + \epsilon_{jln} \omega_n \right) + 3 \left(2 \frac{\partial v_l}{\partial x_m} + \epsilon_{lmn} \omega_n \right) \hat{b}_m \hat{b}_j \right] \\ &= 2\epsilon_{ikl} \hat{b}_k \partial v_j / \partial x_l + (\delta_{in} \delta_{kj} - \delta_{kn} \delta_{ij}) \hat{b}_k \omega_n \\ &\quad + \left[6\epsilon_{ikl} \hat{b}_k \hat{b}_m \partial v_l / \partial x_m + 3(\delta_{im} \delta_{kn} - \delta_{km} \delta_{in}) \hat{b}_k \omega_n \hat{b}_m \right] \hat{b}_j \\ &= 2\hat{\mathbf{b}} \times \nabla \mathbf{v} - \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \mathbf{I} + [6\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \mathbf{v}) - 2\boldsymbol{\omega} + 3\hat{\mathbf{b}} \cdot \boldsymbol{\omega} \hat{\mathbf{b}}] \hat{\mathbf{b}} \Big|_{ij}\end{aligned}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ is the *vorticity* vector, invoking

$$\partial v_l / \partial x_j = \partial v_j / \partial x_l + \epsilon_{jlm} \omega_m$$

at the second line and the property (1.17) at the third line. Hence

$$\begin{aligned}\hat{\mathbf{b}} \times \mathbf{s} \cdot (\mathbf{I} + 3\hat{\mathbf{b}} \hat{\mathbf{b}}) &= \hat{\mathbf{b}} \times \nabla \mathbf{v} - \frac{1}{2} \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \mathbf{I} \\ &\quad + \left[3\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \mathbf{v}) - \boldsymbol{\omega} + \frac{3}{2} \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \hat{\mathbf{b}} \right] \hat{\mathbf{b}} - \frac{1}{3} \nabla \cdot \mathbf{v} \hat{\mathbf{b}} \times \mathbf{I},\end{aligned}$$

with transpose equivalent

$$\begin{aligned}-(\mathbf{I} + 3\hat{\mathbf{b}} \hat{\mathbf{b}}) \cdot \mathbf{s} \times \hat{\mathbf{b}} &= -(\nabla \mathbf{v})^T \times \hat{\mathbf{b}} - \frac{1}{2} \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \mathbf{I} \\ &\quad + \hat{\mathbf{b}} \left[3\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \mathbf{v}) - \boldsymbol{\omega} + \frac{3}{2} \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \hat{\mathbf{b}} \right] + \frac{1}{3} \nabla \cdot \mathbf{v} \mathbf{I} \times \hat{\mathbf{b}}.\end{aligned}$$

Consequently, since $\mathbf{l} \times \hat{\mathbf{b}} - \hat{\mathbf{b}} \times \mathbf{l} = 0$ we obtain

$$\begin{aligned} \mathbf{t}_g &= \frac{\mu}{2|\mathbf{a}|} \left[\hat{\mathbf{b}} \times \mathbf{s} \cdot (\mathbf{l} + 3\hat{\mathbf{b}}\hat{\mathbf{b}}) - (\mathbf{l} + 3\hat{\mathbf{b}}\hat{\mathbf{b}}) \cdot \mathbf{s} \times \hat{\mathbf{b}} \right] \\ &= \frac{\mu}{2|\mathbf{a}|} \left[\hat{\mathbf{b}} \times \nabla \mathbf{v} - (\nabla \mathbf{v})^T \times \hat{\mathbf{b}} \right] - \chi \mathbf{l} + \hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}} \end{aligned}$$

where

$$\chi = \frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \cdot \boldsymbol{\omega} \quad \text{and} \quad \mathbf{A} = \frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \times (3\hat{\mathbf{b}} \cdot \nabla \mathbf{v} + \hat{\mathbf{b}} \times \boldsymbol{\omega}) + \chi \hat{\mathbf{b}},$$

as were prescribed — and the divergence of the first term on the right-hand side immediately yields $-(1/2)\rho \mathbf{v}_* \cdot \nabla \mathbf{v}$, involving the “magnetisation velocity”

$$\mathbf{v}_* = -\frac{1}{\rho} \nabla \times \left(\frac{\mu}{|\mathbf{a}|} \hat{\mathbf{b}} \right)$$

also mentioned in (Q4). This is of course consistent with

$$\begin{aligned} \nabla \cdot \left[\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \times \nabla \mathbf{v} \right] &= -\nabla \cdot \nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \mathbf{v} \right) \\ &\quad + \nabla \cdot \left[\left(\nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right) \mathbf{v} \right] = -\frac{1}{2} \nabla \cdot (\rho \mathbf{v}_* \mathbf{v}) \end{aligned}$$

from the given identity

$$\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \times \nabla \mathbf{v} = -\nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \mathbf{v} \right) + \left[\nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right] \mathbf{v},$$

on noting that $\nabla \cdot (\rho \mathbf{v}_*) = 0$. However, the transpose of this given identity is more useful — i.e.

$$-\frac{\mu}{2|\mathbf{a}|} (\nabla \mathbf{v})^T \times \hat{\mathbf{b}} = \left(\frac{\mu}{2|\mathbf{a}|} \mathbf{v} \hat{\mathbf{b}} \right) \times \overleftarrow{\nabla} + \mathbf{v} \nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right),$$

where the arrow superscript again denotes the leftward gradient operation as before in (Q3) of Sect.2.6, yielding the divergence of the second term

$$\begin{aligned} \nabla \cdot \left(-\frac{\mu}{2|\mathbf{a}|} (\nabla \mathbf{v})^T \times \hat{\mathbf{b}} \right) &= -\nabla \times \left[\mathbf{v} \cdot \nabla \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) + \nabla \cdot \mathbf{v} \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right] \\ &\quad + \nabla \cdot \left[\mathbf{v} \nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right] \end{aligned}$$

such that

$$\begin{aligned}
\nabla \cdot \mathbf{t}_g &= -\frac{1}{2}\rho \mathbf{v}_* \cdot \nabla \mathbf{v} - \nabla \times \left[\mathbf{v} \cdot \nabla \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) + \nabla \cdot \mathbf{v} \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \right] \\
&\quad - \frac{1}{2} \nabla \cdot (\rho \mathbf{v} \mathbf{v}_*) - \nabla \chi + \nabla \cdot (\hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}}) \\
&= -\frac{1}{2}\rho \mathbf{v}_* \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) - (\nabla \cdot \mathbf{v}) \nabla \times \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \\
&\quad - (\nabla \times \mathbf{v}) \cdot \nabla \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) + \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \times \nabla \nabla \cdot \hat{\mathbf{v}} - \frac{1}{2} \nabla \cdot (\rho \mathbf{v} \mathbf{v}_*) \\
&\quad - \nabla \chi + \nabla \cdot (\hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}}) \\
&= -\frac{1}{2}\rho \mathbf{v}_* \cdot \nabla \mathbf{v} + \frac{1}{2} [\mathbf{v} \cdot \nabla (\rho \mathbf{v}_*) + (\rho \nabla \cdot \mathbf{v}) \mathbf{v}_*] \\
&\quad - (\nabla \cdot \mathbf{v}) \times \nabla \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) + \left(\frac{\mu}{2|\mathbf{a}|} \hat{\mathbf{b}} \right) \times \nabla \nabla \cdot \hat{\mathbf{v}} - \frac{1}{2} \nabla \cdot (\rho \mathbf{v} \mathbf{v}_*) \\
&\quad - \nabla \chi + \nabla \cdot (\hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}}) \\
&= -\frac{1}{2}\rho \mathbf{v}_* \cdot \nabla \mathbf{v} + \nabla \times \left[\frac{\mu}{2|\mathbf{a}|} (\nabla \cdot \mathbf{v}) \hat{\mathbf{b}} \right] - \nabla \chi + \nabla \cdot (\hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}}) \\
&= -\rho (\mathbf{v}_* \cdot \nabla) \mathbf{v} - \nabla \chi + 2 \mathbf{B} \cdot \nabla (B^{-1} \mathbf{A}) \\
&\quad - \nabla \times \left[\frac{\mu}{|\mathbf{a}|} \left(\hat{\mathbf{b}} \cdot \nabla \mathbf{v} + \frac{1}{2} (\nabla \cdot \mathbf{v} - 3 \hat{\mathbf{b}} \hat{\mathbf{b}} : \nabla \mathbf{v}) \hat{\mathbf{b}} \right) \right],
\end{aligned}$$

on invoking $\nabla \cdot \mathbf{B} = 0$ so $\nabla \cdot (\hat{\mathbf{b}} \mathbf{A} + \mathbf{A} \hat{\mathbf{b}}) = \nabla \times (\hat{\mathbf{b}} \times \mathbf{A}) + 2 \mathbf{B} \cdot \nabla (B^{-1} \mathbf{A})$.

We recall from Sect.2.9 that the coefficient $\mu/(2|\mathbf{a}|) = mp/(2eB)$, and note that Ramos obtained the above result for $\nabla \cdot \mathbf{t}_g$ from collisionless theory, where p is specified as the perpendicular pressure.¹ As he pointed out, the consequent cancelled part of $\mathbf{v}$ in the advective term of the ion momentum equation (or the equation of motion in an ion-electron plasma) is the “magnetisation velocity” $\mathbf{v}_*$, which only reduces to the legendary “diamagnetic drift velocity” $\mathbf{v}_d = \hat{\mathbf{b}} \times \nabla p / (neB)$ for a constant magnetic field.

Section 3.2

A1) The Bernoulli equation for steady flow under negligible body force (gravity) is

$$\frac{1}{2}v^2 + h = \text{const}, \text{ with } h \equiv \int_{p_0}^p \frac{dp}{\rho} = k \int_{p_0}^p \frac{dp}{p} = k \ln(p/p_0).$$

¹J.J. Ramos (*Physics of Plasmas* **12**, 112301, 2005).

Thus at the inlet

$$\frac{1}{2}v_1^2 + k \ln(p_1/p_0) = \text{const},$$

and at the outlet

$$\frac{1}{2}v_2^2 + k \ln(p_2/p_0) = \text{const},$$

such that

$$\frac{1}{2}(v_1^2 - v_2^2) + k \ln(p_1/p_2) = 0.$$

The steady flow continuity equation

$$\nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{or} \quad \oint_S \rho \mathbf{v} \cdot d\mathbf{S} = 0$$

yields

$$-\rho_1 v_1 \pi \frac{d_1^2}{4} + \rho_2 v_2 \pi \frac{d_2^2}{4} = 0,$$

so that

$$\frac{\rho_1}{\rho_2} = \frac{v_2 d_2^2}{v_1 d_1^2} = \frac{p_1}{p_2} = \exp\left(-\frac{v_1^2 - v_2^2}{2k}\right)$$

and hence the required result.

A2) From (2.35), the rate of change of a volume integral

$$\begin{aligned} \frac{d}{dt} \int_V \left(\frac{1}{2}\rho v^2 + \rho V\right) d\tau &= \int_V \left(\frac{d}{dt}(\frac{1}{2}\rho v^2 + \rho V) + (\frac{1}{2}\rho v^2 + \rho V) \nabla \cdot \mathbf{v}\right) d\tau \\ &= \int_V \left((\frac{1}{2}v^2 + V)\left(\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v}\right) + \rho \frac{d}{dt}(\frac{1}{2}v^2 + V)\right) d\tau \\ &= \int_V \rho \mathbf{v} \cdot \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla(\frac{1}{2}v^2 + V)\right) d\tau \quad \left(\text{assuming } \frac{\partial V}{\partial t} = 0\right). \end{aligned}$$

Now rewriting the equation of motion (3.5) as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2}v^2 + V\right) = \mathbf{v} \times (\nabla \times \mathbf{v}) - \frac{\nabla p}{\rho}$$

yields

$$\rho \mathbf{v} \cdot \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla(\frac{1}{2}v^2 + V)\right) = \rho \mathbf{v} \cdot \left(-\frac{1}{\rho} \nabla p\right),$$

so that

$$\begin{aligned} \frac{d}{dt} \int_V \left(\frac{1}{2}\rho v^2 + \rho V\right) d\tau &= \int_V \rho \mathbf{v} \cdot \left(-\frac{1}{\rho} \nabla p\right) d\tau = - \int_V \mathbf{v} \cdot \nabla p d\tau \\ &= - \int_V [\nabla \cdot (p \mathbf{v}) - p \nabla \cdot \mathbf{v}] d\tau = \int_V p \nabla \cdot \mathbf{v} d\tau - \int_S p \mathbf{v} \cdot d\mathbf{S} \end{aligned}$$

using the Divergence Theorem (1.60).

Recall from Sect.2.5 that $\nabla \cdot \mathbf{v}$ measures the rate of dilatation (i.e. expansion if $\nabla \cdot \mathbf{v} > 0$ or contraction if $\nabla \cdot \mathbf{v} < 0$), and the volume integral on the right-hand side is the corresponding contribution to the energy change. The second integral represents the rate of work done by the surface force (in this case simply the hydrostatic pressure, a particular case of the general form $-\hat{\mathbf{n}} \cdot \mathbf{p}$).

Section 3.3

A1) The velocity of any fluid element rotating with the instantaneous angular velocity $\boldsymbol{\omega}_0$ is $\mathbf{v} = \boldsymbol{\omega}_0 \times \mathbf{r}$, hence

$$\nabla \times \mathbf{v} = \nabla \times (\boldsymbol{\omega}_0 \times \mathbf{r}) = \boldsymbol{\omega}_0 \nabla \cdot \mathbf{r} - \boldsymbol{\omega}_0 \cdot \nabla \mathbf{r} = 2\boldsymbol{\omega}_0$$

identifies the vorticity of any element as twice its angular velocity. Then

$$\begin{aligned} \frac{1}{2}[\boldsymbol{\omega} \times d\mathbf{r}]_i &= \frac{1}{2}\epsilon_{ilm}\omega_l dx_m = \frac{1}{2}\epsilon_{ilm}\epsilon_{ljk}[\boldsymbol{\Omega}]_{jk} dx_m \\ &= \frac{1}{2}\epsilon_{mil}\epsilon_{ljk}[\boldsymbol{\Omega}]_{jk} dx_m = \frac{1}{2}(\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij})\Omega_{jk} dx_m \\ &= \frac{1}{2}(\Omega_{ji} dx_j - \Omega_{ij} dx_j) = dx_j \Omega_{ji} = [d\mathbf{r} \cdot \boldsymbol{\Omega}]_i \end{aligned}$$

A2) Since $\nabla \times \mathbf{v} = 0 \Rightarrow \mathbf{v} = -\nabla \phi$ for irrotational flow, where ϕ is the velocity potential, assuming that its partial derivatives in space and time may be interchanged such that $\partial_t \mathbf{v} = -\partial_t \nabla \phi = -\nabla \partial_t \phi$, equation (3.20) becomes

$$\nabla \left(-\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + h + V \right) = 0$$

where $h \equiv \int dp/\rho$. Thus throughout the flow

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + h + V = F(t),$$

where the arbitrary function of time $F(t)$ can be removed by re-defining the velocity potential to be $\phi' = \phi - \int^t F(u) du$ to obtain

$$-\frac{\partial \phi'}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + h + V = 0,$$

and then dropping the prime. An absolute constant could also be retained on the right-hand side of this result, such that it reduces to the Bernoulli form (3.18) for steady irrotational flow (where $\partial_t \phi = 0$ and $\mathbf{v} = -\nabla \phi$), consistent with $F(t) \equiv \text{constant}$ in the previous equation.

Section 3.4

A1) The volume of water flowing through any pipe cross-section per second must be constant — i.e. from the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$ we have $v_A A = v_B B = k \text{ m}^3$ per second, where v_A and v_B denote the values of the flow speed at cross-sections of area A and B , respectively. Before the tap is opened, the Bernoulli equation along any streamline in the pipe is

$$p_A + \frac{1}{2}\rho v_A^2 + \rho g H_A = p_B + \frac{1}{2}\rho v_B^2 + \rho g H_B ,$$

where p_A and p_B are the respective fluid pressures, and H_A and H_B are the heights of corresponding points on the streamline relative to any arbitrary horizontal reference level (often called the “datum”). The pressure difference

$$p_A - p_B = \frac{1}{2}\rho(v_B^2 - v_A^2) + \rho g(H_B - H_A) ,$$

is eventually equal to the head $\rho g H$ of water in the vertical tube when the tap is open, since p_A may be equated to the outside air pressure on the open pipe and tube. Consequently, assuming $|H_B - H_A| \ll H$ on any streamline, the water may reach a height above the pipe approaching

$$H = \frac{1}{2}\rho(v_B^2 - v_A^2)/(\rho g) = \frac{(k/B)^2 - (k/A)^2}{2g} = \frac{k^2(A^2 - B^2)}{2gA^2B^2} .$$

This result may be used to measure the flow rate k through a pipe (given A , B and H).

A2) Given the water volume remains constant,

$$\frac{4}{3}\pi(S^3 - R^3) = \frac{4}{3}\pi(b^3 - a^3)$$

and hence the first required result. From the spherical symmetry, it is appropriate to adopt spherical coordinates (r, θ, ϕ) but omit any terms involving $\partial/\partial\theta$ and $\partial/\partial\phi$. Thus the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$ becomes

$$\frac{\partial}{\partial r}(r^2 v) = 0 \Rightarrow r^2 v = F(t),$$

where v is the magnitude of the (radial) fluid velocity and $F(t)$ some function of time t . The initial condition $v = \dot{R}$ when $r = R$ yields $F(t) = R^2 \dot{R}$ such that

$$v = \frac{R^2 \dot{R}}{r^2} \Rightarrow \phi = \frac{R^2 \dot{R}}{r} \Rightarrow \frac{\partial \phi}{\partial t} = \frac{1}{r}(2R\dot{R}^2 + R^2\ddot{R}) ,$$

which may be used in the Bernoulli equation for unsteady irrotational flow (the Kelvin theorem implies the flow remains irrotational): thus

$$-\frac{1}{R}(2R\dot{R}^2 + R^2\ddot{R}) + \frac{R^4 \dot{R}^2}{2R^4} = F(t) \text{ since } p = 0 \text{ at } r = R ,$$

$$-\frac{1}{S}(2R\dot{R}^2 + R^2\ddot{R}) + \frac{R^4\dot{R}^2}{2S^4} + \frac{p_0}{\rho} = F(t) \text{ since } p = p_0 \text{ at } r = S,$$

and on subtracting

$$(2R\dot{R}^2 + R^2\ddot{R})\left(\frac{1}{R} - \frac{1}{S}\right) - \frac{R^4\dot{R}^2}{2}\left(\frac{1}{R^4} - \frac{1}{S^4}\right) = -\frac{p_0}{\rho}.$$

Using result the first required result and noting that $d(\dot{R}^2)/dR = 2\ddot{R}$, this equation may be rewritten

$$\frac{d}{dR}\{R^4\dot{R}^2[R^{-1} - (R^3 + b^3 - a^3)^{-\frac{1}{3}}]\} = -\frac{2p_0}{\rho}R^2,$$

which on integration yields the second required result, since $\dot{R} = 0$ when $R = a$. (This result also follows quite readily from energy conservation.)

Section 3.6

A1) We have

$$u - iv = -\frac{\partial\phi}{\partial x} - i\frac{\partial\psi}{\partial x} = -\frac{\partial f}{\partial x} = -\frac{df}{d\zeta}\frac{\partial\zeta}{\partial x} = -\frac{df}{d\zeta};$$

and $|\mathbf{v}|^2 = u^2 + v^2 = (u - iv)(u + iv) = (df/d\zeta)(df/d\zeta)^*$. Consequently:

(a) $-df/d\zeta = Ue^{-i\alpha} = U\cos\alpha - iU\sin\alpha$, so $u = U\cos\alpha$ and $v = U\sin\alpha$, representing a uniform flow of speed U at angle α to the x -axis;

(b) $-df/d\zeta = m/\zeta = (m/r)e^{-i\theta} = (m/r)[\cos\theta - i\sin\theta]$, so $u = (m/r)\cos\theta$ and $v = (m/r)\sin\theta$, representing a flow out from a line source of strength m at the origin — i.e. the flow is radial with velocity $(m/r)\hat{\mathbf{e}}_r$ corresponding to the velocity potential $-m\ln r$, and an outwards flux $(m/r) \times 2\pi r = 2\pi m$.

(c) $-df/d\zeta = i\kappa/\zeta = i(\kappa/r)e^{-i\theta} = (\kappa/r)[i\cos\theta + \sin\theta]$, so $u = (\kappa/r)\sin\theta$ and $v = -(\kappa/r)\cos\theta$, representing a line vortex of strength κ at the origin — i.e. the flow is azimuthal with the velocity $(\kappa/r)\hat{\mathbf{e}}_\theta$, corresponding to a circulation $K = (\kappa/r) \times 2\pi r = 2\pi\kappa$.

A2) The Bernoulli equation along any streamline

$$\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} = \frac{1}{2}U^2 + \frac{p_\infty}{\rho} \quad \Rightarrow \quad p = p_\infty + \frac{1}{2}\rho(U^2 - |\mathbf{v}|^2)$$

for the pressure in the flow. Although the flow is restricted within the upper half of the xy -plane, the complex potential $f(\zeta) = -U(\zeta + a^2/\zeta)$ given by the Milne-Thomson Theorem in the more classical case of flow past a circular cylinder of radius a at the origin may be adopted, since $y = 0$ is then a plane of symmetry. Thus $df/d\zeta = -U(1 - a^2/\zeta^2) = -U(1 - a^2/(ae^{i\theta})^2) = -U(1 - e^{-i2\theta})$ on the semicircular surface $\zeta = ae^{i\theta}$; and since $|\mathbf{v}|^2 = (df/d\zeta)(df/d\zeta)^*$

we have $|\mathbf{v}|^2 = U^2(1 - e^{-i2\theta})(1 - e^{i2\theta}) = U^2[2 - (e^{i2\theta} + e^{-i2\theta})] = 2U^2(1 - \cos 2\theta)$ on that surface. Thus the pressure there is

$$p = p_\infty + \frac{1}{2} \rho U^2 (2 \cos 2\theta - 1) = p_\infty + \frac{1}{2} \rho U^2 (4 \cos^2 \theta - 3) ,$$

and integrating over this semicircular surface $r = a$, $0 < \theta < \pi$ yields the consequent vertical force per unit length on the cylinder due to the flow

$$\begin{aligned} \int_0^\pi (p - p_\infty) \sin \theta a d\theta &= -\frac{1}{2} \rho a U^2 \int_0^\pi (-\sin \theta) (4 \cos^2 \theta - 3) d\theta \\ &= -\frac{1}{2} \rho a U^2 \left[\frac{4}{3} \cos^3 \theta - 3 \cos \theta \right]_0^\pi = -\frac{5}{3} \rho a U^2 \end{aligned}$$

— i.e. the lift on the cylinder has magnitude $(5/3)\rho a U^2$ per unit length. The pressure difference $p - p_\infty$ is the appropriate input here, since the downward pressure force is balanced by the base support of the cylinder when there is no flow.

The horizontal force per unit length on the cylinder is

$$\int_0^\pi p \cos \theta a d\theta = \int_0^\pi (\cos \theta) \left[p_\infty + \frac{1}{2} \rho U^2 (1 - 4 \sin^2 \theta) \right] a d\theta = 0 ,$$

so according to ideal (inviscid) theory there is no drag.

In passing, we note that planes of symmetry are exploited in the method of images mentioned in the text. Thus the image of any line source or line vortex in a flow bounded by an infinite plane is a reflection through that plane that is then notionally removed, analogous to the adoption of the complex potential in the solution to the flow past a semicircular cylinder here. The method of images may also be exploited in three dimensions, such as for point sources or sinks.

A3) Conformal transformations preserve the Laplace equation. We consider the suggested bilinear transformation, to render the annular region of cross-section between *concentric* circular cylinders where the velocity potential is the simple “polar harmonic” form $\phi(r) = c_1 \ln r + c_0$ involving the radial coordinate r and constants c_0 and c_1 , and readily obtain the result from the related complex potential $f(\zeta) = \phi(x, y) + i\psi(x, y)$ and relevant bilinear transformation as follows.

The corresponding complex potential is $f(\zeta) = c_1 \text{Log } \zeta + c_0$, where Log denotes the principal value of the complex logarithmic function. The conjugate stream function $\psi(\zeta) = c_1 \Im(\text{Log } \zeta) = c_1 \theta$ (where $-\pi < \theta \leq \pi$) may be verified from the Cauchy-Riemann equations, using the well known coordinate transformation $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(y/x)$. The bilinear conformal transformation

$$w = \frac{\zeta - \alpha}{\alpha^* \zeta - 1} , \quad |\alpha| < 1 ,$$

maps the unit disc onto the unit disk — i.e. for $|\zeta| = 1$ we have

$$|\zeta - \alpha| = |\zeta^* - \alpha^*| = |\zeta||\zeta^* - \alpha^*| = |\zeta\zeta^* - \alpha^*\zeta| = |1 - \alpha^*\zeta| = |\alpha^*\zeta - 1|$$

such that $|w| = 1$. The origin $\zeta = 0$ maps to $w = -\alpha$ where $|w| = |\alpha| < 1$, so any point on the unit disc in the ζ -plane (bounded by the circle $|\zeta| = 1$) is mapped to the unit disc in the w -plane. Similarly, for the circle $|\zeta - 2/5| = 2/5$

$$\begin{aligned} |\zeta - \alpha| &= |\zeta^* - \alpha^*| = \frac{5}{2} \left| \zeta - \frac{2}{5} \right| |\zeta^* - \alpha^*| \\ &= \left| \frac{5}{2} \left(\zeta - \frac{2}{5} \right) \left(\zeta^* - \frac{2}{5} \right) + \frac{5}{2} \left(\zeta - \frac{2}{5} \right) \left(\frac{2}{5} - \alpha^* \right) \right| \\ &= \left| \frac{2}{5} + \zeta - \frac{5}{2} \alpha^* \zeta - \frac{2}{5} + \alpha^* \right| \\ &= |\alpha| \left| \left(\frac{5}{2} - \frac{1}{\alpha^*} \right) \zeta - 1 \right| = |\alpha| |\alpha^* \zeta - 1| \end{aligned}$$

provided $5/2 - 1/\alpha^* = \alpha^*$ or $\alpha^{*2} - 5\alpha^*/2 + 1 = 0$, corresponding to $\alpha^* \in \{1/2, 2\}$. Indeed, choosing $\alpha = 1/2$ such that $|w| = 1/2$ maps the circle $|\zeta - 2/5| = 2/5$ onto $|w| = 1/2$ under the bilinear transformation

$$w = f(\zeta) = \frac{2\zeta - 1}{\zeta - 2}$$

— i.e. onto an inner circle *concentric* to the outer circle $|w| = 1$. From the complex potential $f(w) = c_1 \text{Log } w + c_0$ for the region between these two concentric circles in the w -plane, we therefore immediately deduce that the complex potential in the region between the two non-concentric circles in the z -plane is

$$f(\zeta) = c_1 \text{Log} \left(\frac{2\zeta - 1}{\zeta - 2} \right) + c_0,$$

such that the velocity potential is

$$\phi(x, y) = \Re \left[c_1 \text{Log} \left(\frac{2\zeta - 1}{\zeta - 2} \right) + c_0 \right] = c_1 \ln \left(\frac{2\zeta - 1}{\zeta - 2} \right) + c_0$$

where $\zeta = x + iy$.

In passing, we note that the constants c_0, c_1 are of course generally determined by the boundary conditions at the two cylindrical surfaces — in particular, we have (3.6) such that $\partial\phi/\partial n \equiv \hat{\mathbf{n}} \cdot \nabla\phi = -\mathbf{v}_s$ in the context of potential flow.

Section 3.7

A1) (a) We have

$$\begin{aligned}\mathbf{v} &= -\nabla\phi = -\left(\hat{\mathbf{e}}_r \frac{\partial}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}\right) \left[a |\mathbf{v}_0| \theta - |\mathbf{v}_\infty| \left(r + \frac{a^2}{r}\right) \cos \theta \right] \\ &= |\mathbf{v}_\infty| \left(1 - \frac{a^2}{r^2}\right) \cos \theta \hat{\mathbf{e}}_r - \left[\frac{a}{r} |\mathbf{v}_0| + |\mathbf{v}_\infty| \left(1 + \frac{a^2}{r^2}\right) \sin \theta \right] \hat{\mathbf{e}}_\theta ,\end{aligned}$$

and streamlines defined by $d\mathbf{r} = dr \hat{\mathbf{e}}_r + r d\theta \hat{\mathbf{e}}_\theta = \lambda \mathbf{v}$ or $dr = \lambda v_r$ and $r d\theta = \lambda v_\theta$, whence

$$\frac{dr}{d\theta} = -\frac{|\mathbf{v}_\infty| r (1 - a^2/r^2) \cos \theta}{a |\mathbf{v}_0|/r + |\mathbf{v}_\infty| (1 + a^2/r^2) \sin \theta} .$$

When $|\mathbf{v}_\infty| = 0$ the streamlines are obviously concentric circles $r = \text{const}$, and the two requested cases also have analytical solutions as follows.

(i) $\mathbf{v}_0 = 0$.

$$\frac{dr}{d\theta} = \frac{r(a^2 - r^2)}{(a^2 + r^2) \tan \theta} \Rightarrow \int_{r_1}^{r_2} \frac{a^2 + r^2}{r(a^2 - r^2)} dr = \int_{\theta_1}^{\theta_2} \frac{d\theta}{\tan \theta} \Rightarrow \frac{r_2 r_1^2 - a^2}{r_1 r_2^2 - a^2} = \frac{\sin \theta_2}{\sin \theta_1}$$

in which we can set $\theta_1 = \pi/2$ and take r_1 to be the point of closest approach of the streamline to $r = a$. Thus for any given r_1 we have $\sin \theta_2$ as a function of r_2 , and hence also $\cos \theta_2$, so we can compute the streamlines as the locus of points with $x = r_2 \cos \theta_2$ and $y = r_2 \sin \theta_2$. The graphical solution is left to the student as a computer programming exercise. (A suitable computer software package could be used, and of course the streamlines could also be sketched by the method of isoclines.)

(ii) $\mathbf{v}_0 = \mathbf{v}_\infty$.

$$\frac{dr}{d\theta} = \frac{r^2(a^2 - r^2) \cos \theta}{a + r(a^2 + r^2) \sin \theta} \quad \text{or} \quad \frac{du}{dr} - \frac{a^2 + r^2}{r(a^2 - r^2)} u = \frac{a}{r^2(a^2 - r^2)}$$

on setting $u = \sin \theta$. This is a linear ordinary differential equation in $u(r)$ with integrating factor $(a^2 - r^2)/r$, whence

$$\sin \theta_2 = \frac{r_2}{r_1} \frac{r_1^2 - a^2}{r_2^2 - a^2} \sin \theta_1 + \frac{a}{2} \frac{r_2}{r_2^2 - a^2} \left(\frac{1}{r_2^2} - \frac{1}{r_1^2} \right)$$

and we may proceed as before.

The flow is compatible with a right cylinder of circular cross-section with radius $r = a$, aligned with the z -axis. Case (i) represents the flow of a uniform stream (corresponding to $r \rightarrow \infty$) past a cylinder; and in case (ii) there is also circulation around it, as found below.

(b) Substituting the above velocity components immediately renders the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v} = \hat{\mathbf{e}}_z/r [\partial(rv_\theta)/\partial r - \partial v_r/\partial \theta] = 0$, which is of course

consistent with $\boldsymbol{\omega} = \nabla \times (-\nabla \phi) = 0$. Similarly, the circulation about $r = 2a$, $0 \leq \theta < 2\pi$ is

$$K = \oint_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} v_\theta|_{r=2a} 2a d\theta = -2\pi a |\mathbf{v}_0| ,$$

consistent with

$$K = - \oint_C d\mathbf{r} \cdot \nabla \phi = - \oint_C d\phi = -[\phi(2a, 2\pi) - \phi(2a, 0)] .$$

A circle of radius a centred on $r = 3a$, $\theta = 0$ or $(x, y) = (3a, 0)$ is the locus of points with position vector $\mathbf{r} = (3 + \cos \alpha) a \hat{\mathbf{i}} + \sin \alpha \hat{\mathbf{j}}$, if we use α to denote the angle to any point on the circle relative to the x -axis. With this or some other geometric construction, the circulation could be evaluated from the velocity field, but it again follows immediately from the velocity potential. Indeed, the circle of radius a begins and ends at the same point $r = 4a$, $\theta = 0$ so the circulation is zero — as it must be around any reducible curve in the region $r > a$, consistent with Stokes Theorem (1.57) since $\nabla \times \mathbf{v} = \nabla \times (-\nabla \phi) = 0$.

(c) The incompressibility constraint

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} = 0$$

follows immediately on substituting for v_r and v_θ from above. Consequently, the density is $\rho_0 = \text{const}$ everywhere, since $\mathbf{v} \cdot \nabla \rho = 0 \Rightarrow d\mathbf{r} \cdot \nabla \rho = 0$ follows from (3.1) in the steady flow.

(d) From (3.60), the pressure is $-\frac{1}{2}\rho|\mathbf{v}|^2$ — i.e.

$$p(r, \theta) = -\frac{\rho}{2} \left[|\mathbf{v}_\infty| \left(1 - \frac{a^2}{r^2} \right) \cos \theta \right]^2 + \left[\frac{a}{r} |\mathbf{v}_0| + |\mathbf{v}_\infty| \left(1 + \frac{a^2}{r^2} \right) \sin \theta \right]^2 ,$$

which produces a vertical force per unit length of the cylinder

$$\begin{aligned} \int_0^{2\pi} p(a, \theta) \sin \theta a d\theta &= -\frac{1}{2} \rho a \int_0^{2\pi} (|\mathbf{v}_0| + 2|\mathbf{v}_\infty| \sin \theta)^2 \sin \theta a d\theta \\ &= -\frac{1}{2} \rho a \int_0^{2\pi} [|\mathbf{v}_0|^2 + 4|\mathbf{v}_0||\mathbf{v}_\infty| \sin \theta + 4|\mathbf{v}_\infty|^2 \sin^2 \theta] \sin \theta d\theta \\ &= -\frac{1}{2} \rho a \int_0^{2\pi} [2|\mathbf{v}_0||\mathbf{v}_\infty| (1 + \cos 2\theta)] d\theta \\ &= -2\pi \rho a |\mathbf{v}_0| |\mathbf{v}_\infty| , \end{aligned}$$

a lift of the magnitude $\rho |\mathbf{v}_\infty| K$ given by the Kutta and Joukovskii Theorem mentioned in Sect.3.6, and in the horizontal direction

$$\begin{aligned} \int_0^{2\pi} p(a, \theta) \cos \theta a d\theta &= -\frac{1}{2} \rho a \int_0^{2\pi} (|\mathbf{v}_0| + 2|\mathbf{v}_\infty| \sin \theta)^2 \sin \theta a d\theta \\ &= -\frac{1}{2} \rho a \int_0^{2\pi} [|\mathbf{v}_0|^2 + 4|\mathbf{v}_0||\mathbf{v}_\infty| \sin \theta + 4|\mathbf{v}_\infty|^2 \sin^2 \theta] \cos \theta d\theta \\ &= 0, \end{aligned}$$

so once again ideal (inviscid) theory predicts there is no drag.

A2) Let us adopt spherical coordinates with the centre of the sphere at the origin, and the originating flow of velocity $\mathbf{U}$ is parallel to the $\theta = 0$ axis. The flow is axisymmetric about this axis, such that the velocity potential $\phi(r, \theta)$ satisfies the Laplace equation

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) \\ &= r^2 \frac{\partial^2 \phi}{\partial r^2} + 2r \frac{\partial \phi}{\partial r} + \cot \theta \frac{\partial \phi}{\partial \theta} + \frac{\partial^2 \phi}{\partial \theta^2} \\ &= 0. \end{aligned}$$

This partial differential equation is to be solved subject to boundary conditions (1) $\mathbf{v} \rightarrow \mathbf{U}$ as $|\mathbf{r}| \rightarrow \infty$; and (2) $\hat{\mathbf{n}} \cdot \nabla \phi = \partial \phi / \partial r = U \cos \theta$ at $r = a$.

The elementary technique of separation of variables where $\phi(r, \theta) = R(r) \Theta(\theta)$ renders

$$\frac{r^2 R'' + 2r R'}{R} = -\frac{\Theta'' + \cot \theta \Theta'}{\Theta} = \text{const.}$$

Consequently, the well-behaved solution at $\theta = 0$ and $\theta = \pi$ corresponds to $\Theta(\theta) = P_n(\cos \theta)$ (a Legendre polynomial) when the constant is $n(n+1)$ for any integer n . The general solution of $r^2 R'' + 2r R' - n(n+1)R = 0$ is $R(r) = A_n r^n + B_n / r^{n+1}$, where A_n and B_n are associated constants, since setting $R(r) = r^\alpha$ into this ordinary differential equation gives $\alpha = n$ and $\alpha = -(n+1)$. The boundary conditions then require $n = 1$, and finally render the solution for the velocity potential

$$\phi(r, \theta) = -U \left(r + \frac{a^3}{2r^2} \right) \cos \theta.$$

Hence

$$\nabla \phi = \left(\hat{\mathbf{e}}_r \frac{\partial \phi}{\partial r} + \hat{\mathbf{e}}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) = -U \left[1 - \left(\frac{a}{r} \right)^3 \right] \cos \theta \hat{\mathbf{e}}_r + U \left[1 + \frac{1}{2} \left(\frac{a}{r} \right)^3 \right] \sin \theta \hat{\mathbf{e}}_\theta,$$

hence from (3.60) the pressure variation at the surface of the sphere $r = a$ (due to the steady flow and neglecting gravity) is

$$p(a) = -\frac{9}{8} \rho U^2 \sin^2 \theta .$$

Integrating over the surface of the sphere, we have the the axial force

$$\begin{aligned} \int_S \hat{\mathbf{k}} \cdot [-p(a) \hat{\mathbf{e}}_{\mathbf{r}}] dS &= - \int_0^{2\pi} \int_0^\pi p(a) \cos \theta a^2 \sin \theta d\theta d\phi \\ &= \int_0^{2\pi} \int_0^\pi \frac{9}{8} \rho U^2 \sin^3 \theta a^2 \cos \theta d\theta d\phi \\ &= 0, \end{aligned}$$

so the ideal theory once again predicts there is no drag due to the flow.

In passing, we note that the pressure could have been obtained (within a constant) from the steady flow Bernoulli equation along a streamline — cf. (A2) of Sect.3.6.

Section 3.9

A1) Recall from (Q1) of Sect.3.6 that a line vortex of strength κ at the origin $r = 0$ has the azimuthal flow speed $v_\theta(r) = \kappa/r$, with a singularity such that the swirling flow rapidly increases as $r \rightarrow 0$. For two line vortices of strength κ at $x = \pm a$ respectively, the corresponding flow contributions are

$$v_\theta(r) = \frac{\kappa}{|\mathbf{r} \mp a\hat{\mathbf{i}}|} .$$

The fluid velocity due to one vortex produces a translational velocity $\kappa/(2a)$ of the other vortex in the direction of the y -axis, while the first has a relative translational velocity due to the second of the same magnitude but in the opposite (negative y -axis) direction, since the line vortices move with the fluid. The pair therefore rotates about the z -axis, in a circle of radius a , as they move with any background flow. There is observational evidence that hurricanes sometimes do move in pairs, but unfortunately the prediction of their joint path or subsequent divergent motion remains an imprecise science!

A2) Consider any line vortex at point P in the upper row $y = b$, say the one initially at $x = Na$. At point P , the fluid motion due to any neighbouring line vortex in the upper row to the right is cancelled by the opposing contribution from another in the upper row to the left — i.e. the nett flow at any line vortex due to all the others *in the same row* is zero — but any pair of line vortices in the other row does have a nett effect. Consider any two vortices at the points $Q_{1,2}$ where $x = [N \pm (n + \frac{1}{2})]$ say, in the lower

row $y = -b$. Although the y -components of the fluid velocity due to that pair at $Q_{1,2}$ cancel at P , the x -components reinforce to produce (recalling $K = 2\pi\kappa$)

$$2 \frac{K}{2\pi\sqrt{4b^2 + (n + \frac{1}{2})^2 a^2}} \frac{2b}{\sqrt{4b^2 + (n + \frac{1}{2})^2 a^2}} = \frac{2Kb}{\pi[4b^2 + (n + \frac{1}{2})^2 a^2]}$$

at P . Summing over all such pairs yields the drift speed

$$\frac{2K}{\pi a} \sum_{n=0}^{\infty} \frac{4(b/a)}{16(b/a)^2 + (2n+1)^2} = \frac{K}{2a} \tanh\left(\frac{2\pi b}{a}\right)$$

in the x -direction. The same is true for any line vortex in the upper row, and also for any line vortex in the lower row.

This system constitutes the Karman vortex street, which arises in the flow past a “blunt” body.

A3) The fluid velocity at any point $P(x, y, z)$ due to a line vortex element at $(\xi, 0, 0)$ of length $d\xi$ on the x -axis is $(u, v, 0)$, where

$$u = -\frac{\kappa y d\xi}{(x - \xi)^2 + y^2}, \quad v = \frac{\kappa(x - \xi) d\xi}{(x - \xi)^2 + y^2}.$$

Integration over $(-\infty, \infty)$ yields the result for the sheet vortex.

Section 3.10

A1) The result stated in (Q5) of Sect.1.6 (with $h_1 = 1, h_2 = r, h_3 = r \sin \theta$), or alternatively (0.7) in (A2) of Sect.1.5, provides the vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}$ with only a ϕ -component

$$\begin{aligned} \omega_\phi &= \frac{1}{r} \left(\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right) \\ &= \frac{1}{r} \left[\frac{\partial}{\partial r} \left(-\frac{1}{\sin \theta} \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial \theta} \left(\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \right) \right] \\ &= -\frac{1}{r \sin \theta} \left[\frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \psi \\ &= -\frac{1}{r \sin \theta} \mathcal{L} \psi \quad \text{say.} \end{aligned}$$

as expected in the axisymmetric flow. Then setting $\nabla \times \boldsymbol{\omega}$ in (3.80) produces

$$\begin{aligned} \nabla p &= -\frac{\mu}{r \sin \theta} \left[\frac{\partial}{\partial \theta} \left(-\frac{1}{r} \mathcal{L} \psi \right) \right] \hat{\mathbf{e}}_r + \frac{\mu}{r} \left[\frac{\partial}{\partial r} \left(-\frac{1}{\sin \theta} \mathcal{L} \psi \right) \right] \hat{\mathbf{e}}_\theta \\ &= \frac{\mu}{r^2 \sin \theta} \frac{\partial(\mathcal{L} \psi)}{\partial \theta} \hat{\mathbf{e}}_r - \frac{\mu}{r \sin \theta} \frac{\partial(\mathcal{L} \psi)}{\partial r} \hat{\mathbf{e}}_\theta, \end{aligned}$$

whence the biharmonic equation (3.85) from

$$\frac{\partial}{\partial \theta} \frac{\partial p}{\partial r} = \frac{\partial}{\partial r} \frac{\partial p}{\partial \theta}.$$

Since this last step reflects taking the curl of (3.80), equation (3.86) obviously also directly follows from (3.85), on applying the Laplacian operator ∇^2 stated in (Q5) of Sect.1.6 on $\boldsymbol{\omega} = \omega_\phi \hat{\mathbf{e}}_\phi$ where $\hat{\mathbf{e}}_\phi = -\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{j}}$.

A2) Substituting the Stokes solution (3.88) gives

$$\begin{aligned} \mathcal{L}\psi &= \left[\frac{\partial^2}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \left[\frac{1}{4} U \left(2r^2 - 3ar + \frac{a^3}{r} \right) \sin^2 \theta \right] \\ &= \frac{3Ua}{2r} \sin^2 \theta, \end{aligned}$$

so integration of $\partial p / \partial r$ from the above answer yields the pressure variation

$$p_\infty - p(r) = \int_r^\infty \frac{\mu}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\frac{3Ua}{2r} \sin^2 \theta \right) dr = \frac{3\mu Ua}{r^2} \cos \theta$$

between the pressure far away from the sphere and at radius r , respectively.

Noting that $\hat{\mathbf{n}} = \hat{\mathbf{e}}_{\mathbf{r}}$ on the sphere, we have

$$\hat{\mathbf{e}}_{\mathbf{r}} \cdot [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] = \hat{\mathbf{e}}_{\mathbf{r}} \cdot \nabla \mathbf{v} + \nabla (\mathbf{v} \cdot \hat{\mathbf{e}}_{\mathbf{r}}) - (\nabla \hat{\mathbf{e}}_{\mathbf{r}}) \cdot \mathbf{v} = \frac{\partial \mathbf{v}}{\partial r} + \nabla v_r - \hat{\mathbf{e}}_\theta \frac{v_\theta}{r}$$

since $\nabla \hat{\mathbf{e}}_{\mathbf{r}} = (\hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \hat{\mathbf{e}}_\phi \hat{\mathbf{e}}_\phi)/r$. Consequently, substituting for p from above in

$$\hat{\mathbf{e}}_{\mathbf{r}} \cdot (-p \mathbf{I} + \mu [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]) = \left[-p + 2\mu \frac{\partial v_r}{\partial r} \right] \hat{\mathbf{e}}_{\mathbf{r}} + \left[\mu r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{\mu}{r} \frac{\partial v_r}{\partial \theta} \right] \hat{\mathbf{e}}_\theta$$

yields the viscous force per unit area on the surface of the sphere at $r = a$:

$$\hat{\mathbf{e}}_{\mathbf{r}} \cdot \mathbf{P}|_{r=a} = \left[-p_\infty + \frac{3\mu U}{2a} \cos \theta \right] \hat{\mathbf{e}}_{\mathbf{r}} - \left[\frac{3\mu U}{2a} \sin \theta \right] \hat{\mathbf{e}}_\theta = -p_\infty \hat{\mathbf{e}}_{\mathbf{r}} + \frac{3\mu}{2a} \mathbf{U}.$$

The Stokes drag on the sphere in the axial flow is thus

$$\begin{aligned} D &= \int_0^{2\pi} \int_0^\pi \hat{\mathbf{k}} \cdot (\hat{\mathbf{e}}_{\mathbf{r}} \cdot \mathbf{P}|_{r=a}) a^2 \sin \theta d\theta d\phi \\ &= \int_0^{2\pi} \int_0^\pi \left(-p_\infty \cos \theta + \frac{3\mu}{2a} U \right) a^2 \sin \theta d\theta d\phi \\ &= 6\pi \mu a U. \end{aligned}$$

A3) Inserting the specified ordering forms into equations (3.80) and (3.81),

$$\begin{aligned} &\left[\varepsilon \left(\hat{\mathbf{i}} \frac{\partial}{\partial X} + \hat{\mathbf{j}} \frac{\partial}{\partial Y} \right) + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right] (p^{(0)} + \varepsilon p^{(1)} + \dots) \\ &= \mu \left[\varepsilon^2 \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \right) + \frac{\partial^2}{\partial z^2} \right] (\varepsilon \mathbf{v}^{(1)} + \varepsilon^2 \mathbf{v}^{(2)} + \dots) \end{aligned}$$

$$\text{and} \quad \left[\varepsilon \left(\hat{\mathbf{i}} \frac{\partial}{\partial X} + \hat{\mathbf{j}} \frac{\partial}{\partial Y} \right) + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right] \cdot (\varepsilon \mathbf{v}^{(1)} + \varepsilon^2 \mathbf{v}^{(2)} + \dots) = 0 ,$$

such that the first equation above yields

$$\text{O}(1) \quad \frac{\partial p^{(0)}}{\partial z} = 0 \quad \Rightarrow \quad p^{(0)}(X, Y)$$

$$\begin{aligned} \text{O}(\varepsilon) \quad \left(\hat{\mathbf{i}} \frac{\partial}{\partial X} + \hat{\mathbf{j}} \frac{\partial}{\partial Y} \right) p^{(0)} + \hat{\mathbf{k}} \frac{\partial p^{(1)}}{\partial z} &= \mu \frac{\partial^2 \mathbf{v}^{(1)}}{\partial z^2} \\ \Rightarrow \quad \nabla_{\perp} p^{(0)} &= \mu \frac{\partial^2 \mathbf{v}_{\perp}^{(1)}}{\partial z^2} \quad \text{and} \quad \frac{\partial p^{(1)}}{\partial z} = \mu \frac{\partial^2 v_z^{(1)}}{\partial z^2} , \end{aligned}$$

and the second equation above yields

$$\text{O}(\varepsilon) \quad \frac{\partial v_z^{(1)}}{\partial z} = 0$$

such that $v_z^{(1)} = 0$ from the no penetration boundary condition $\hat{\mathbf{n}} \cdot \mathbf{v} = 0$ at $z = \pm a$; and consequently

$$\frac{\partial p^{(1)}}{\partial z} = 0 \quad \Rightarrow \quad p^{(1)}(X, Y) ,$$

$$\text{and} \quad \nabla_{\perp} p^{(0)} = \mu \frac{\partial^2 \mathbf{v}_{\perp}^{(1)}}{\partial z^2} \quad \Rightarrow \quad \mathbf{v}^{(1)} \equiv \mathbf{v}_{\perp}^{(1)} = -\frac{a^2 - z^2}{2\mu} \nabla_{\perp} p^{(0)}$$

on applying the “no slip” boundary condition $\hat{\mathbf{n}} \times \mathbf{v} = 0$ at $z = \pm a$.

It also immediately follows that $\nabla_{\perp}^2 p^{(0)} = 0$, since $\nabla_{\perp} \cdot \mathbf{v}_{\perp} = 0$.

Section 3.11

A1) Let us adopt Cartesian coordinates with origin at a fixed point on the lower plate, the x -axis along the flow direction and the y -axis perpendicular to the parallel plates. The steady unidirectional flow $\mathbf{v} = v(y)\hat{\mathbf{i}}$ satisfies the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$; and the components of the Navier-Stokes equation of motion (3.62) for flow in the xy -plane are

$$\begin{aligned} \frac{\partial p}{\partial x} &= \rho g \sin \alpha + \mu \frac{d^2 v}{dy^2} , \\ \frac{\partial p}{\partial y} &= -\rho g \sin \alpha , \end{aligned}$$

viewing the parallel plates as sloping downward left to right. The second equation gives

$$\frac{\partial^2 p}{\partial x \partial y} = 0 \Rightarrow \frac{\partial p}{\partial x} = -G , \text{ the negative downslope applied pressure gradient .}$$

Thus the first equation becomes

$$\frac{d^2v}{dy^2} = -\frac{1}{\mu}(G + \rho g \sin \alpha)$$

with general solution

$$v(y) = -\frac{1}{2\mu}(G + \rho g \sin \alpha) y^2 + c_1 y + c_2 .$$

The “no slip” boundary conditions $y(0) = y(d) = 0$ determine the integration constants c_1 and c_2 , and hence the solution

$$v(y) = \frac{G + \rho g \sin \alpha}{2\mu} y(d - y) ,$$

which produces the the flow per unit width on integrating from $y = 0$ to $y = d$.

A2) The axial component of the Navier-Stokes equation of motion (3.62) for the unsteady unidirectional flow along the cylinder is

$$\rho \frac{\partial v}{\partial t} - G = \mu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) ,$$

subject to the homogeneous “no slip” boundary condition $v(a, t) = 0$ for $t > 0$ and initial condition $v(r, 0) = 0$ for $0 \leq r \leq a$. The nonhomogeneous partial differential equation can be solved in a standard way — viz. by a dependent variable transformation $w(r, t) = v_0(r) - v(r, t)$, where

$$v_0(r) = \frac{G}{4\mu}(a^2 - r^2)$$

obtained from the ordinary differential equation

$$-G = \mu \left(\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right)$$

subject to $v(r, 0) = 0$ is the well known parabolic steady profile for the flow. The detailed solution of the resulting homogeneous partial differential equation

$$\frac{\partial w}{\partial t} = \frac{\mu}{\rho} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)$$

subject to the nonhomogeneous conditions

$$w(a, t) = 0 \text{ for } t > 0 \quad \text{and} \quad w(r, 0) = \frac{G}{4\mu}(a^2 - r^2) \text{ for } 0 \leq r \leq a$$

is quite straightforward — cf. G.K. Batchelor, “*An Introduction to Fluid Mechanics*” (2000), where there are various other exact viscous flow cases presented, including for example the smoothing out of a velocity discontinuity at a plane.

Steady flow through a circular cylinder due to a pressure gradient, originally investigated by Hagen and Poiseuille in the first half of the eighteenth century, provided an early model for blood flow and also for pipe flow in engineering practice. The steady rate of flow of course immediately follows by integrating the steady velocity field over a cross-section of the circular cylinder.

The sudden time-dependent flow solution has the whole fluid accelerated as G/ρ initially, but the viscous shear is progressively transmitted into the flow — so that the central portion of fluid moving with velocity increasing as Gt/ρ narrows, as the steady Hagen-Poiseuille flow is approached Batchelor *ibid.* However, the asymptotic steady state may be unstable, as Reynolds first observed. The unidirectional flow persists at sufficiently low flow speeds, but becomes turbulent at higher load speeds — i.e. above a critical value of the Reynolds number Re we introduced in Sect.3.12 — and this eventually stimulated some of the earliest hydrodynamic stability theory — P.G. Drazin and W.H. Reid, “*Hydrodynamic Stability*” (2004).

A3) The steady azimuthal flow field $\mathbf{v}(r) = v(r)\hat{\mathbf{e}}_\theta$ provides an advective term $\mathbf{v} \cdot \nabla \mathbf{v} = -\hat{\mathbf{e}}_r v^2/r$, balancing the pressure gradient $\nabla p = \hat{\mathbf{e}}_r dp/dr$ in the radial component of the Navier-Stokes equation of motion (i.e. $v^2/r = dp/dr$), and the azimuthal component of (3.62) reduces to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right) v = \frac{d}{dr} \left(\frac{d}{dr} + \frac{1}{r} \right) v = 0 ,$$

with general solution $v(r) = c_1 r + c_2/r$ in $a \leq r \leq b$ subject to the “no slip” boundary conditions $v(a) = a|\boldsymbol{\omega}_0|$ and $v(b) = 0$. Consequently

$$v(r) = -|\boldsymbol{\omega}_0| \left(\frac{a^2}{b^2 - a^2} r + \frac{1}{b^{-2} - a^{-2}} \frac{1}{r} \right) = |\boldsymbol{\omega}_0| \frac{a^2}{r} \frac{b^2 - r^2}{b^2 - a^2} ,$$

such that the angular velocity $\boldsymbol{\omega} = \mathbf{r} \times \mathbf{v}/r^2 = \omega(r)\hat{\mathbf{e}}_z$ where $\omega(r) = v(r)/r$. The force $-\nabla p + \mu \nabla^2 \mathbf{v}$ in the Navier-Stokes equation (3.62) produces the moment $\mu \mathbf{r} \times \nabla^2 \mathbf{v} = \mu \mathbf{r} \times (\nabla \times \boldsymbol{\omega}) = -(\mu d\omega/dr)\hat{\mathbf{e}}_\theta$ on a fluid particle at radius r , and hence the torque per unit length of cylinder

$$\mu |\boldsymbol{\omega}_0| \int_{r=a}^b \int_{\theta=0}^{2\pi} \frac{d\omega}{dr} r dr d\theta = 2\pi \mu |\boldsymbol{\omega}_0| \int_{r=a}^b \frac{2a^2 b^2}{b^2 - a^2} \frac{1}{r^2} dr = -4\pi \mu \frac{a^2 b^2}{b^2 - a^2} |\boldsymbol{\omega}_0| .$$

This question may also be answered by dotting the conservation of angular momentum equation (2.30) with $\hat{\mathbf{e}}_z$, and noting that $\mathbf{p} = p\mathbf{l} - \mu[\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$ in the Navier-Stokes equation (since $\nabla \cdot \mathbf{v} = 0$).

Couette flow is another classical area of hydrodynamic stability, with theoretical and experimental investigations of ideal and viscous flow first carried out by Rayleigh and G.I. Taylor, as discussed in Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover). In particular, Taylor showed

that the above viscous flow is stable only if $|\boldsymbol{\omega}_0|$ is sufficiently small, and there is a similar stability condition when the outer cylinder rotates and the inner cylinder is fixed.

Section 3.13

A1) Including the inertia term, (3.84) becomes $\rho \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = \mu \nabla^2 \nabla \times \mathbf{v}$, or $-\rho \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \mu \nabla^2 \boldsymbol{\omega}$ since $\mathbf{v} \cdot \nabla \mathbf{v} = \nabla(v^2/2) - \mathbf{v} \times (\nabla \times \mathbf{v})$.

$$\begin{aligned} \text{Now } \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) &= \mathbf{v} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{v} \\ &= \left(v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} \right) \left(-\frac{1}{r \sin \theta} \mathcal{L}\psi \hat{\mathbf{e}}_\phi \right) \\ &\quad + \frac{1}{r \sin \theta} (\mathcal{L}\psi) \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (v_r \hat{\mathbf{e}}_r + v_\theta \hat{\mathbf{e}}_\theta), \end{aligned}$$

on adopting $\boldsymbol{\omega} = \omega \hat{\mathbf{e}}_\phi$ from (A1) of Sect.3.10. Recalling that $\partial \hat{\mathbf{e}}_r / \partial \phi = \sin \theta \hat{\mathbf{e}}_\phi$ and $\partial \hat{\mathbf{e}}_\theta / \partial \phi = \cos \theta \hat{\mathbf{e}}_\phi$, we have

$$\begin{aligned} \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) &= \hat{\mathbf{e}}_\phi \left(\frac{1}{r^2 \sin \theta} v_r - \frac{1}{r \sin \theta} v_r \frac{\partial}{\partial r} \right. \\ &\quad \left. + \frac{1}{r^2 \sin \theta} \cot \theta v_\theta - \frac{1}{r^2 \sin \theta} v_\theta \frac{\partial}{\partial \theta} \right. \\ &\quad \left. + \frac{1}{r^2 \sin^2 \theta} (v_r \sin \theta + v_\theta \cos \theta) \right) \mathcal{L}\psi, \end{aligned}$$

and therefore (3.92) on rendering $-\rho \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = \mu \nabla^2 \boldsymbol{\omega}$ into dimensionless form where $\text{Re} = \rho U a / \mu$ is the associated Reynolds number.

(Introduce the dimensionless forms $\nabla^* = a \nabla$ and $\mathbf{v}^* = \mathbf{v} / U$ and then omit the asterisks.)

A2) We have

$$\frac{\partial^2}{\partial r^{(-1)2}} \mathcal{L}\psi_1 = \exp\left(\frac{r^{(-1)} \cos \theta}{2}\right) \left(\frac{\partial^2}{\partial r^{(-1)2}} + \cos \theta \frac{\partial}{\partial r^{(-1)}} + \frac{1}{4} \cos^2 \theta \right) \phi,$$

$$\begin{aligned} \frac{\sin \theta}{r^{(-1)2}} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \mathcal{L}\phi \right) &= \exp\left(\frac{r^{(-1)} \cos \theta}{2}\right) \\ &\times \left(\frac{1}{r^{(-1)2}} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^{(-1)2}} \frac{\partial}{\partial \theta} - \frac{\sin \theta}{r^{(-1)}} \frac{\partial}{\partial \theta} + \frac{1}{4} \sin^2 \theta \right) \phi \end{aligned}$$

such that

$$\begin{aligned} &\left(\mathcal{L} - \cos \theta \frac{\partial}{\partial r^{(-1)}} + \sin \theta \frac{\partial}{\partial \theta} \right) \mathcal{L}\psi_1 \\ &= \exp\left(\frac{r^{(-1)} \cos \theta}{2}\right) \left(\frac{\partial^2}{\partial r^{(-1)2}} + \frac{1}{r^{(-1)2}} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^{(-1)2}} \frac{\partial}{\partial \theta} - \frac{1}{4} \right) \phi = 0 \end{aligned}$$

or $(\mathcal{L} - 1/4)\phi = 0$. Then setting $\phi = f(r^{(-1)}) \sin^2 \theta$ produces

$$\left(\mathcal{L} - \frac{1}{4}\right)\phi = f'' \sin^2 \theta + \frac{2}{r^{(-1)2}}(\cos^2 \theta - \sin^2 \theta)f - \frac{2 \cos^2 \theta}{r^{(-1)}}f - \frac{1}{4} \sin^2 \theta f ,$$

whence

$$f'' - \left(\frac{2}{r^{(-1)2}} + \frac{1}{4}\right)f = 0 .$$

The result (3.102) may be verified by direct substitution into this differential equation, and likewise the particular solution (3.101) by direct substitution into $\mathcal{L}\psi_1 = C(1 + 2/r^{(-1)}) \exp[-(r^{(-1)}/2)(1 - \cos \theta)]$.

A3) To match over intermediate distances from the gap, express the outer asymptotic solution in terms of the inner variables X and Z for comparison with the inner asymptotic solution rewritten in terms of the outer variables r and z . Thus since $x = \exp[-\pi(r - \lambda_1) - iz] = 1 - \pi\epsilon(X - iZ) + O(\epsilon^2)$, in terms of the inner variables

$$\begin{aligned} \phi_{\text{outer}} &= \frac{1}{2}\lambda_1 \ln(\lambda_2/\lambda_1) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} R_n(\lambda_1) - \frac{1}{\pi} \left(1 + \frac{\epsilon X}{\lambda_1}\right)^{-1/2} \\ &\quad \times \ln |\pi\epsilon(X - iZ)| - \frac{4}{8\pi^2\lambda_1} \Re\{1 - \pi\epsilon(X - iZ) \\ (0.10) \quad &\quad + \pi\epsilon(X - iZ) \ln[\pi\epsilon(X - iZ)]\} + O(\epsilon^2) \\ &= \frac{1}{2}\lambda_1 \ln(\lambda_2/\lambda_1) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} R_n(\lambda_1) \\ &\quad - \frac{1}{\pi} \ln(\pi\epsilon\sqrt{X^2 + Y^2}) - \frac{1}{2\pi^2\lambda_1} + O(\epsilon) , \end{aligned}$$

on noting that two terms in $\epsilon \ln \epsilon$ cancel. (Such a cancellation often occurs.) For the inner expansion near the gap, we have $(r - \lambda_1)/\epsilon \simeq e^u \sin v$ and $z/\epsilon \simeq \cos v$, so that in terms of the outer variables

$$\phi_{\text{inner}} = V_0 - \frac{1}{\pi} \ln \left(\frac{2}{\epsilon} \sqrt{(r - \lambda_1)^2 + z^2} \right) + O(\epsilon) .$$

The inner and outer expansions therefore match over intermediate distances provided

$$V_0 = \frac{1}{2}\lambda_1 \ln(\lambda_2/\lambda_1) + \frac{1}{\pi} \ln \left(\frac{2}{\pi\epsilon} \right) - \frac{1}{2\pi^2\lambda_1} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} R_n(\lambda_1) .$$

Section 4.1

A1) Since $kx - \omega(k)t \simeq k_0 - \omega(k_0)t + (k - k_0)\xi$ where $\xi = x - c_g(k_0)t$, substitution into the wave form integral produces the quoted first result.

Then substituting $a(k) = a_0 \exp[-\lambda(k - k_0)^2]$ and writing $\kappa = k - k_0$, we have

$$\begin{aligned} F(\xi) &\simeq a_0 \int_{\mathcal{D}} \exp[-\lambda\kappa^2 + i\xi\kappa] d\kappa \\ &= a_0 \int_{\mathcal{D}} \exp[-\lambda(\kappa - i\xi/(2\lambda))^2 - \xi^2/(4\lambda)] d\kappa \\ &= a_0 \exp[-\xi^2/(4\lambda)] \int_{\mathcal{D}} (-\lambda\eta^2) d\eta \end{aligned}$$

where $\eta = \kappa - i\xi/(2\lambda)$, whence for $\lambda \gg 1$

$$F(\xi) \simeq a_0 \sqrt{\frac{\pi}{\lambda}} \exp \left[-\frac{(x - c_g(k_0)t)^2}{4\lambda} \right] \exp[i(k_0x - \omega(k_0)t)]$$

on noting that $\int_{\mathcal{D}} (-\lambda\eta^2) d\eta \simeq \int_{-\infty}^{\infty} (-\lambda\eta^2) d\eta = \sqrt{\pi/\lambda}$.

The large parameter λ and group speed $c_g(k_0)$ define the envelope of the localised wave packet around the wave number k_0 .

This calculation resembles the classical asymptotic evaluation of an integral of form $\int_{\mathcal{D}} f(x) \exp(-\lambda x) dx$ for parameter $\lambda \gg 1$, originally due to Laplace. Note also that the exponential Fourier form is often convenient in the analysis, but ultimately it is the real part of the solution that represents the relevant physical quantity.

A2) Noting that $f(k_x/k, k_y/k, k_z/k) = f(\hat{\mathbf{e}}_k)$ as $\mathbf{k}/k = \hat{\mathbf{e}}_k$, the group velocity

$$\begin{aligned} \mathbf{c}_g &= \frac{\partial}{\partial \mathbf{k}} [kf(\hat{\mathbf{e}}_k)] \\ &= \hat{\mathbf{e}}_k f(\hat{\mathbf{e}}_k) + [\mathbf{I} - \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k] \cdot \frac{\partial f}{\partial \hat{\mathbf{e}}_k} = \mathbf{c} + [\mathbf{I} - \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k] \cdot \frac{\partial f}{\partial \hat{\mathbf{e}}_k} \end{aligned}$$

because

$$\begin{aligned} \frac{\partial k}{\partial \mathbf{k}} &= \frac{1}{2k} \frac{\partial k^2}{\partial \mathbf{k}} = \frac{1}{2k} \frac{\partial (\mathbf{k} \cdot \mathbf{k})}{\partial \mathbf{k}} = \frac{1}{k} \mathbf{I} \cdot \mathbf{k} = \frac{\mathbf{k}}{k} = \hat{\mathbf{e}}_k, \\ \frac{\partial \hat{\mathbf{e}}_k}{\partial \mathbf{k}} &= \frac{\partial}{\partial \mathbf{k}} \left(\frac{\mathbf{k}}{k} \right) = \frac{1}{k} \mathbf{I} - \frac{\mathbf{k}}{k^2} \frac{\partial k}{\partial \mathbf{k}} = \frac{1}{k} \mathbf{I} - \frac{\mathbf{k}}{k^2} \hat{\mathbf{e}}_k, \end{aligned}$$

where we have used $\partial \mathbf{k} / \partial k = \mathbf{I}$ analogous to $\nabla \mathbf{r} = \mathbf{I}$. Consequently, $\hat{\mathbf{e}}_k \cdot \mathbf{c}_g = \hat{\mathbf{e}}_k \cdot \mathbf{c} = c$ so that the projection of $\mathbf{c}_g$ is $\hat{\mathbf{e}}_k \hat{\mathbf{e}}_k \cdot \mathbf{c}_g = \mathbf{c}$ at the base of the right-angled triangle, with $\mathbf{c}_g$ the vector as the hypotenuse.

Section 4.2

A1) We have

$$\Delta \mathbf{k}_1 \cdot \Delta \mathbf{r}_2 = \Delta \mathbf{r}_1 \cdot \nabla \mathbf{k} \cdot \Delta \mathbf{r}_2 = \Delta \mathbf{r}_2 \cdot \nabla \mathbf{k} \cdot \Delta \mathbf{r} = \Delta \mathbf{k}_2 \cdot \Delta \mathbf{r}_1.$$

For the solution by characteristics, this must be propagated in time by the ray equation. Define $f_{12} = \Delta \mathbf{k}_1 \cdot \Delta \mathbf{r}_2 - \Delta \mathbf{k}_2 \cdot \Delta \mathbf{r}_1$ and use (4.18) and (4.19): thus

$$\begin{aligned} \frac{d}{dt} \Delta \mathbf{k}_i &= -\Delta \mathbf{r}_i \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{r}} - \Delta \mathbf{k}_i \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{r}} , \\ \frac{d}{dt} \Delta \mathbf{r}_i &= \Delta \mathbf{r}_i \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{k}} + \Delta \mathbf{k}_i \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{k}} , \\ \text{so } \frac{d}{dt} f_{12} &= -\Delta \mathbf{r}_1 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{r}} \cdot \Delta \mathbf{r}_2 - \Delta \mathbf{k}_1 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{r}} \cdot \Delta \mathbf{r}_2 + \Delta \mathbf{r}_2 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{k}} \cdot \Delta \mathbf{k}_1 \\ &\quad + \Delta \mathbf{k}_2 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{k}} \cdot \Delta \mathbf{k}_1 + \Delta \mathbf{r}_2 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{r}} \cdot \Delta \mathbf{r}_1 + \Delta \mathbf{k}_2 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{r}} \cdot \Delta \mathbf{r}_1 \\ &\quad - \Delta \mathbf{r}_1 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{r} \partial \mathbf{k}} \cdot \Delta \mathbf{k}_2 - \Delta \mathbf{k}_1 \cdot \frac{\partial^2 \Omega}{\partial \mathbf{k} \partial \mathbf{k}} \cdot \Delta \mathbf{k}_2 \\ &= 0 . \end{aligned}$$

Section 4.3

A1) We have $x' = x - Vt$ so $v' = v - V$, hence $kx - \omega t = k(x' + Vt) - \omega t = kx' - (\omega - kV)t'$ such that $k = k'$ and $\omega' = \omega - k'V$. Consequently, we obtain $\omega - kv = \omega' + k'V - k'(v' + V) = \omega' - k'v'$ in the x -direction.

A2) Assuming ρ and p are initially constant in the ideal fluid at rest ($\mathbf{v} = 0$) and subsequent purely axial flow in the thin tube, dependent only on the coordinate x along the tube in addition to the time t , the fundamental linearised perturbation equations from (3.5) and (3.3) reduce to

$$\begin{aligned} \frac{\partial v_1}{\partial t} &= \frac{1}{\rho} \frac{\partial p_1}{\partial x} \\ \frac{\partial p_1}{\partial t} &= -\gamma p \frac{\partial v_1}{\partial x} , \end{aligned}$$

producing the one-dimensional wave equation

$$\frac{\partial^2 v_1}{\partial t^2} + c_s^2 \frac{\partial^2 v_1}{\partial x^2} = 0$$

where $c_s = \sqrt{\gamma p / \rho}$ is the sound speed. Thus the general solution may be written $v_1 = a \exp[i(kx - \omega t)] = (A \cos kx + B \sin kx) e^{-i\omega t}$ (where $c_s = \omega / k$).

(a) When the pipe is connected end to end to form a toroidal cavity, we have $v_1(x) = v_1(x + \ell) \forall x$ such that $\exp(ikx) = \exp[ik(x + \ell)]$ — i.e. $\exp(ik\ell) = 1$ and hence $k = 2\pi n / \ell$ for $n = 1, 2, \dots$. The lowest mode corresponds to $n = 1$, with wavelength $\lambda = 2\pi / k = \ell$ (the length of the pipe).

(b) When the pipe is closed at both ends (with “stops”), applying the condition $v_1 = 0$ at $x = 0$ and $x = \ell$ implies $A = 0$ and then $\sin k\ell = 0$, so that $k = n\pi / \ell$ for $n = 1, 2, \dots$. The lowest mode again corresponds to $n = 1$,

with wavelength $\lambda = 2\pi/k = 2\ell$ (*twice* the length of the pipe).

Historical notes:

(i) Rayleigh invoked the velocity potential for an inviscid fluid in discussing sound propagation in pipes with both stops and open ends, on adopting the single space coordinate x for the thin tube and anticipating that “when the cross-section is small and does not vary in area, straightness is not a matter of importance” (i.e. under the simple mathematical model represented above); and he noted the analogy with normal mode theory [cf. Glossary Section “Normal Mode”] for a stretched string, where the phase speed $\sqrt{T/\rho}$ (with T the string tension and ρ its mass per unit length) replaces the sound speed c_s (cf. *Theory of Sound* Volume II, Second Edition, Macmillan 1896). The tone from an organ-pipe or a wind instrument such as a flute or recorder partly depends on whether its end is open or closed, and notes from a stringed instrument are mathematical relatives!

(ii) Kirchhoff analysed sound propagation in a viscous fluid contained in a rigid cylindrical tube of circular cross-section, allowing also for heat conduction within the ideal gas (cf. *ibid* pages 319 – 326). His mathematical model essentially supplemented (3.1) with the compressible viscous equation of motion

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p = \mu \left(\nabla^2 \mathbf{v} + \frac{1}{3} \nabla \nabla \cdot \mathbf{v} \right)$$

that we obtain from (3.2) and (3.4), and a simplified energy equation. His exact travelling wave solution for small perturbations (e.g. on adopting Cauchy boundary conditions at the tube wall) involves somewhat complicated expressions, but yields simpler approximations for wide and thin tubes and in the small viscosity limit. Incidentally, Rayleigh also observed that there is no dissipation in the thin tube limit, when “the temperature of the solid walls controls that of the included gas”.²

²Viscous dissipation has since been included in the energy equation — thus from (2.58) we have

$$\frac{3}{2} \left(\frac{dp}{dt} - \frac{\gamma p}{\rho} \frac{d\rho}{dt} \right) = \mu \left[\frac{1}{2} (\nabla \mathbf{v}) + \frac{1}{2} (\nabla \mathbf{v})^T - \frac{2}{3} \nabla \cdot \mathbf{v} \mathbf{I} \right] : \nabla \mathbf{v} - \nabla \cdot \mathbf{q} .$$

However, the historical Fourier law of heat conduction $\mathbf{q} = -\lambda \nabla T$ rendering $\nabla \cdot \mathbf{q} = -\lambda \nabla^2 T$ has generally been retained, although that is a substantial simplification of (2.86) when $\mathbf{a}_0 = 0$. The Kirchhoff solution is for a flow with radial and axial components — i.e. where the fluid velocity $\mathbf{v}(r, z, t) = v_r(r, z, t) \hat{\mathbf{e}}_r + v_z(r, z, t) \hat{\mathbf{e}}_z$ in a central cylindrical coordinate system with z -axis in the direction of the sound propagation along the tube, such that

$$\nabla \mathbf{v} = \frac{\partial v_r}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_r + \frac{\partial v_z}{\partial r} \hat{\mathbf{e}}_r \hat{\mathbf{e}}_z + \frac{v_r}{r} \hat{\mathbf{e}}_\theta \hat{\mathbf{e}}_\theta + \frac{\partial v_r}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_r + \frac{\partial v_z}{\partial z} \hat{\mathbf{e}}_z \hat{\mathbf{e}}_z .$$

A3) The fundamental linearised perturbation equations from (3.1), (3.5) and (3.3) are

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \rho \nabla \cdot \mathbf{v}_1 &= 0 \\ \rho \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 &= -\rho \nabla V_1 \\ \frac{\partial p_1}{\partial t} &= -\gamma p \nabla \cdot \mathbf{v}_1 ,\end{aligned}$$

now supplemented by $\nabla^2 V_1 = 4\pi G \rho_1$.

Eliminating $\nabla \cdot \mathbf{v}_1$ between the first and third equations produces

$$\frac{\partial p_1}{\partial t} = c_s^2 \frac{\partial \rho_1}{\partial t} \Rightarrow p_1 = c_s^2 \rho_1$$

where $c_s = \sqrt{\gamma p / \rho}$ is the sound speed; and the divergence of the second equation together with the fourth then gives

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \nabla^2 \rho_1 + (4\pi G \rho) \rho_1 .$$

Thus in the absence of self-gravitation we have the classical hyperbolic wave equation; but with self-gravitation we have the additional term $4\pi G \rho$, and consequently $\omega^2 = k^2 c_s^2 - 4\pi G \rho$ for the Fourier form $C \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ where C is a constant.

This result produces the Jeans criterion for the break-up of uniform cosmic gas into proto-galaxies due to self-gravitation — viz. there is instability for wave numbers $k < k_c$, where $k_c = \sqrt{4\pi G \rho / c_s^2}$ is the critical wave number.

Section 4.5

A1) Ocean-going vessels mainly encounter predominantly gravity waves with wavelengths of several to many metres (“swells”) generated by storms, although earlier much shorter wavelength ripples of predominantly capillary waves (cf. Sect.4.6) are also observed when a freshening wind blows over smooth water. Ocean depths are typically thousands of metres such that $|kH| \gg 1$ and hence $|\tanh kH| \simeq 1$, so that (4.55) reduces to $\omega^2 \simeq gk$ and therefore $c(k) \simeq \sqrt{g/k}$ for typical dispersive ocean waves.

A tsunami is a gravity wave generated by a sudden local water displacement due to an earthquake. Their very long wavelengths range up to a hundred kilometres, so in contrast they barely disperse and their phase speeds are hundreds of metres per second — i.e. up to around a thousand kilometres per hour. The devastating tsunami of 26 December 2004 travelled from its source near Sumatra across the Indian Ocean to the east coast of Africa in just over seven hours. The amplitude of a tsunami is usually quite small at

sea, where its extremely long wavelength can mean its crests arrive up to an hour apart, and a seafarer may not even notice its passage — but in coastal shallows the incoming waves slow down and pile up into massive destructive walls of water many metres high. Tsunamis and other long wavelength (infragravity) waves that also propagate long distances have recently been considered the likely trigger for ice shelf collapse in Antarctica.

A2) The dispersion relation is $(\omega - \mathbf{k} \cdot \mathbf{v})^2 = gk$ in the channel, allowing for the phase shifted frequency $\omega - \mathbf{k} \cdot \mathbf{v}$ due to the background flow velocity $\mathbf{v}(y) = Vy\hat{\mathbf{i}}$, and also deep water as in (Q1). Thus adopting the positive square root to render $\omega > 0$ in the channel (for $y < 0$), we have $\omega = \mathbf{k} \cdot \mathbf{v} + \sqrt{gk} = k_x Vy + \sqrt{gk}$. Noting $\partial/(\partial \mathbf{k}) = \hat{\mathbf{i}} \partial/(\partial k_x) + \hat{\mathbf{j}} (\partial k_y) + \hat{\mathbf{k}} (\partial k_z)$ analogous to ∇ , the ray equations (4.18) and (4.19) are

$$\begin{aligned} \frac{d\mathbf{k}}{dt} &= -\frac{\partial \omega}{\partial \mathbf{r}} = -k_x V \hat{\mathbf{e}}_y, \\ \frac{d\mathbf{r}}{dt} &= \frac{\partial \omega}{\partial \mathbf{k}} = \mathbf{v} + \frac{1}{2} \sqrt{\frac{g}{k}} \hat{\mathbf{e}}_k, \end{aligned}$$

where $\hat{\mathbf{e}}_k = \mathbf{k}/k$. Thus on the rays we have

$$\frac{dk_x}{dt} = 0 \implies k_x = \text{const},$$

independent of y since the rays traverse the y -direction for $y < y_c$; and

$$\frac{dk_y}{dt} = -k_x V = \text{const} < 0 \implies k_y = k_0 - k_x V t,$$

choosing the initial time such that the ray crosses $y = 0$ at $t = 0$ when $k_y = k_0$ (the value at the still water at $t = 0$). Consequently,

$$\begin{aligned} \frac{dy}{dt} &= \frac{\sqrt{g}}{2} \frac{k_y}{k^{3/2}} \\ &= \frac{\sqrt{g}}{2} \frac{k_0 - k_x V t}{[k_x^2 + (k_0 - k_x V t)^2]^{3/4}}, \end{aligned}$$

whence

$$\begin{aligned} y &= \frac{\sqrt{g}}{2} \int_0^t \frac{k_0 - k_x V t'}{[k_x^2 + (k_0 - k_x V t')^2]^{3/4}} dt' \\ &= \frac{\sqrt{g}}{2} \frac{1}{k_x V} \int_{k_0}^{k_y} \frac{(-k'_y)}{[k_x^2 + (k'_y)^2]^{3/4}} dk'_y, \end{aligned}$$

such that the sign reversal (where $k_y = 0$) occurs at

$$\begin{aligned} y_c &= \frac{\sqrt{g}}{2} \int_0^{k_0} 2 \frac{d}{dk'_y} [k_x^2 + (k'_y)^2]^{1/4} dk'_y \\ &= \frac{\sqrt{g}}{k_x V} [(k_x^2 + k_0^2)^{1/4} - k_x^{1/2}]. \end{aligned}$$

A3) The initial condition

$$\zeta_t(x, 0) = \int_{-\infty}^{\infty} i\Omega(k)[-a(k) + b(k)] \exp(ikx) dk = 0$$

implies $b(k) = a(k)$ for all $k \neq 0$, and we can ignore the isolated point $k = 0$ of measure zero or simply set $b(0) = a(0)$ for convenience. Thus $\zeta(x, 0) = \int_{-\infty}^{\infty} 2a(k) \exp(ikx) dk = \zeta_0(x)$, whence $2a(k) = \tilde{\zeta}_0(k)$ and (4.57). The alternative solution form follows immediately; and with $\tilde{\zeta}_0(k)$ symmetric, the propagation to the right is

$$\zeta_R(x, t) = \int_0^{\infty} \tilde{\zeta}_0(k) \cos[i(kx - \Omega t)] dk = \Re \left(\int_0^{\infty} \tilde{\zeta}_0(k) \exp[i(kx - \Omega t)] dk \right).$$

A4) The Fourier coefficient in (4.57) is

$$\tilde{\zeta}_0(k) = \frac{1}{2\pi} \int_{-R}^R Q \exp(-ikx) dk = \frac{Q}{2\pi} \left[\frac{e^{-ikx}}{-ik} \right]_{-R}^R = \frac{Q}{\pi k} \sin(kR)$$

and hence the quoted result, yielding propagation to the right as

$$\zeta_R(x, t) = \Re \left(\frac{Q}{\pi} \int_0^{\infty} \frac{\sin(kR)}{k} \exp[i(kx - \Omega t)] dk \right).$$

For small ripples on a deep pond, we have $kH \gg 1$ such that

$$\Omega(k) = \sqrt{gk}, \quad \Omega'(k) = \frac{1}{2} \sqrt{\frac{g}{k}}, \quad \Omega''(k) = -\frac{1}{4} \sqrt{\frac{g}{k^3}},$$

whence $h(k) = kx - \Omega(k)t$ has a stationary point where

$$h'(k) = x - \Omega'(k)t = x - \frac{1}{2} \sqrt{\frac{g}{k}} t = 0 \quad \text{--- i.e. at } k_0(x, t) = \frac{gt^2}{4x^2}.$$

Consequently,

$$h(k_0) = \frac{gt^2}{4x^2} x - \sqrt{g \frac{gt^2}{4x^2}} t = -\frac{gt^2}{4x}$$

$$\text{and} \quad h''(k_0) = -\Omega''(k_0)t = \frac{1}{4} \sqrt{\frac{g}{(gt^2/4x^2)^3}} t = \frac{2x^3}{gt^2},$$

such that the asymptotic result

$$\begin{aligned} \zeta_R(x, t) &\simeq \Re \left(\frac{Q}{\pi} \sqrt{\frac{2\pi}{2x^3/gt^2}} \frac{\sin(gt^2 R/4x^2)}{gt^2/4x^2} \exp[-i(\frac{gt^2}{4x} - \frac{\pi}{4})] \right) \\ &= \frac{4Q}{t} \sqrt{\frac{x}{\pi g}} \sin \left(\frac{gt^2 R}{4x^2} \right) \cos \left(\frac{gt^2}{4x} - \frac{\pi}{4} \right) \end{aligned}$$

closely approximates the eventual wave form propagating away to the right. Such asymptotic results typically clarify wave form features that tend to be hidden in exact Fourier integral solutions, although the FFT (fast Fourier

transform) numerical algorithm may also be invoked to produce descriptive graphical solutions — cf. V.A. Squire, R.J. Hosking, A.D. Kerr and P.J. Langhorne, *Moving Load on Ice Plates* (1996), J. Lighthill, *Waves in Fluids* (2001).

Section 4.6

A1) On the understanding that $Dk^2 + \rho g/k^2 > N$, from the dispersion relation (4.61) we immediately obtain the phase speed $c(k) = \omega/k = \pm\Omega(k)$ where

$$\Omega(k) = \sqrt{\frac{Dk^3 - Nk}{\rho} + \frac{g}{k}},$$

for $kH \gg 1$ such that $\tanh kH \simeq 1$. The positive branch $c(k) = \Omega(k)$ has a stationary point when

$$\frac{dc}{dk} = \frac{1}{2} \left(\sqrt{\frac{Dk^3 - Nk}{\rho} + \frac{g}{k}} \right)^{-1/2} \left(\frac{3Dk^2 - N}{\rho} - \frac{g}{k^2} \right) = 0,$$

or where $3Dk^4 - Nk^2 - \rho g = 0$ such that

$$k^2 = \frac{N \pm \sqrt{N^2 + 12\rho g D}}{6D} \quad \text{or} \quad k^2 = \sqrt{\frac{\rho g}{3D}} (\sinh \epsilon + \cosh \epsilon) = \sqrt{\frac{\rho g}{3D}} e^\epsilon,$$

on substituting $N = \sqrt{12\rho g D} \sinh \epsilon$ and choosing the positive sign on the square root. Since $c(k) \rightarrow \infty$ when $k \rightarrow 0$ and also when $k \rightarrow \infty$, this stationary point is a minimum at the phase number

$$k_{\min} = \left(\frac{\rho g}{3D} \right)^{1/4} e^{\epsilon/2}.$$

The corresponding minimum phase speed is thus

$$\begin{aligned} c_{\min} &= \sqrt{\left(\frac{Dg^3}{27\rho} \right)^{1/4} [e^{3\epsilon/2} - 3(e^{3\epsilon/2} - e^{-\epsilon/2}) + 3e^{-\epsilon/2}]} \\ &= \sqrt{\left(\frac{Dg^3}{27\rho} \right)^{1/4} e^{-\epsilon/2} [e^{2\epsilon} - 3(e^{2\epsilon} - 1) + 3]} \quad , \end{aligned}$$

yielding the desired result

$$c_{\min} = 2 \left(\frac{Dg^3}{27\rho} \right)^{1/8} e^{-\epsilon/4} \sqrt{(3 - e^{2\epsilon})/2}, \quad \text{where } \epsilon = \sinh^{-1} \sqrt{\frac{N}{12\rho g D}}.$$

Thus in the deep water limit where $\tanh(k_{\min}H) \simeq 1$, the reduced dispersion relation defines the minimum phase speed $c_{\min}$ at the wave number $k_{\min}$ as

above; and it is notable that the minimum phase speed

$$c_{\min} = 2 \left(\frac{Dg^3}{27\rho} \right)^{1/8}$$

in the absence of in-plane stress (i.e. when $\epsilon = 0$) is lowered by a compressive stress $N > 0$ such that $0 < \epsilon < (\ln 3)/2$. In passing, we also note that there is a correction that incorporates the explicit dependence of $c_{\min}$ on the fluid depth according to (4.61), if the assumption $\tanh(k_{\min}H) \simeq 1$ is inappropriate.

A2) At short wavelengths where $Tk^2/(\rho g) \gg 1$ and $\tanh(kH) \simeq 1$, the dispersion relation (4.62) yields $\omega = \pm\Omega(k)$ where $\Omega(k) \simeq \sqrt{T/\rho}|k|^{3/2}$, so that we have $c(k) \simeq \sqrt{T|k|/\rho} < c_g(k) \simeq (3/2)\sqrt{T|k|/\rho}$. At longer wavelengths and typical values of the surface tension parameter T , we obtain the dispersion relation (4.55) for gravity waves — where $c > c_g$ on water of finite depth H , as discussed in Sect.4.5. Thus there are predominantly capillary waves propagated ahead of a surface source moving relative to the water, and predominantly gravity waves trailing behind.

Section 4.7

A1) The vorticity $\boldsymbol{\omega} = \nabla \times U(z) \hat{\mathbf{i}} = dU/dz \hat{\mathbf{j}} = (U_1 - U_2) \delta(z) \hat{\mathbf{j}}$, corresponding to a nonzero vortex sheet at $z = 0$.

Consequently, since

$$\int_{z=-b/2}^{z=b/2} \delta(z) dz = [H(z)]_{z=-b/2}^{z=b/2} = 1,$$

we have

$$\begin{aligned} \int_R \boldsymbol{\omega} \cdot d\mathbf{S} &= \int_{x=-l/2}^{l/2} \int_{z=-b/2}^{z=b/2} (U_1 - U_2) \delta(z) dx dz \\ &= (U_1 - U_2) \int_{x=-l/2}^{l/2} dx \int_{z=-b/2}^{z=b/2} \delta(z) dz \\ &= (U_1 - U_2) l, \end{aligned}$$

equal to

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_{x=-l/2}^{l/2} (U_1 - U_2) dx = (U_1 - U_2) l.$$

Section 4.11

A1) (a) The entropy for an adiabatic process is $s = s_0 + c_v \ln(pp^{-\gamma})$, where s_0 denotes a reference value and c_v is the coefficient of specific heat at constant volume, so $\llbracket s \rrbracket = c_v \llbracket \ln(pp^{-\gamma}) \rrbracket$ is the entropy jump across an isolated shock. Consequently, the second law of thermodynamics (the entropy of an isolated

system can never decrease) implies (4.132).

At the end of Sect.2.7, we mentioned that the adiabatic equation of state (2.59) is more precisely called isentropic — and now observe that $p\rho^{-\gamma} = \text{const}$ following the motion implies $\llbracket s \rrbracket = 0$ across the shock, corresponding to the equality in (4.132). On the other hand, unless the shock is weak there is significant dissipation in the transition layer, and the inequality in (4.132) corresponds to a resulting entropy increase ($\llbracket s \rrbracket > 0$) across the shock — i.e. increased disorder, according to the famous epitaph on Ludwig Boltzmann's tombstone! Incidentally, although the brief elementary discussion of shock structure in Sect.4.12 does not reflect the viscous dissipation term on the right-hand side of (2.58), Burger's equation (4.135) does model the viscous dissipation in (2.56) due to the shear viscosity in a fluid.

(b) On substituting $p/\rho = (c_p - c_v)T$ from the well known ideal gas equation (where $\gamma = c_p/c_v$ is the ratio of specific heats c_p and c_v and T is the temperature), the energy jump condition (4.131) immediately yields

$$\left[\left[\frac{1}{2}v^2 + c_p T \right] \right] = 0 \quad \implies \quad T_2 - T_1 = \frac{1}{2c_p} (v_1^2 - v_2^2) > 0 ,$$

on assuming that the specific heat at constant pressure c_p is identical either side of the shock.

Section 5.5

A1) With $\mathbf{B} = \nabla \times (\psi \hat{\mathbf{k}})$ the resistive equation of magnetic induction (5.30)

$$\nabla \times \left(\frac{\partial \psi}{\partial t} \hat{\mathbf{k}} \right) = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

corresponds to (with no prevailing electric field)

$$\frac{\partial \psi}{\partial t} \hat{\mathbf{k}} = \mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B} .$$

Since

$$\mathbf{v} \times \mathbf{B} = v(r) \hat{\mathbf{e}}_\theta \times \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{\mathbf{e}}_r - \frac{\partial \psi}{\partial r} \hat{\mathbf{e}}_\theta \right) = -\frac{v(r)}{r} \frac{\partial \psi}{\partial \theta} \hat{\mathbf{e}}_z$$

$$\text{and} \quad \nabla \times \mathbf{B} = \nabla \times (\nabla \times (\psi \hat{\mathbf{e}}_z)) = -\nabla^2 \psi \hat{\mathbf{e}}_z ,$$

we have the equation

$$\frac{\partial \psi}{\partial t} + \frac{v}{r} \frac{\partial \psi}{\partial \theta} = \eta \nabla^2 \psi .$$

In the steady state $\partial \psi / \partial t = 0$, so for $r < a$

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{k}{\eta r^2} \frac{\partial \psi}{\partial \theta} = 0 \quad \implies \quad r^2 \frac{d^2 f}{dr^2} + r \frac{df}{dr} - \left(1 + \frac{ik}{\eta} \right) f = 0$$

on writing $\psi = f(r) \exp(i\theta)$, such that $f(r) = c_1 r^\lambda + C_2 r^{-\lambda}$ where $\lambda = \sqrt{1 + ik/\eta}$, and we require $c_2 = 0$ for the solution to be well behaved (non-singular) at $r = 0$ — i.e. we have $\psi(r, \theta) = c_1 r^\lambda \exp(i\theta)$ when $r < a$. For $r > a$, the solution of $\nabla^2 \psi = 0$ is $\psi(r, \theta) = (c_3 r + c_4/r) \exp(i\theta)$ where $c_3 = B_0$ such that $\psi \rightarrow B_0 r \exp(i\theta)$ so $\text{Im} \psi \rightarrow B_0 r \sin \theta$ as $r \rightarrow \infty$, analogous to the appropriate velocity potential in uniform planar flow from left to right within a minus sign. (The uniform velocity field $\mathbf{v} = U\hat{\mathbf{i}}$ corresponds to the stream function $\psi = -Ur \sin \theta$ in cylindrical coordinates — cf. the imaginary part of the complex potential $f(\zeta) = -U\zeta$ when $\alpha = 0$ in (Q1) of Sect.3.6.) The values and first derivatives of the flux function ψ for $\mathbf{B}$ must match at $r = a$, so

$$c_1 a^\lambda = B_0 a + \frac{c_4}{a} \quad \text{and} \quad \lambda c_1 a^{\lambda-1} = B_0 - \frac{c_4}{a^2}$$

such that $c_1 = 2B_0 a^{1-\lambda}/(1+\lambda)$. Consequently, for $r < a$ we have

$$\psi(r, \theta) = \frac{2B_0 a}{1+\lambda} \left(\frac{r}{a}\right)^\lambda \exp(i\theta).$$

Then writing $\lambda = \alpha + i\beta$ produces

$$1 + \lambda = |1 + \lambda| \exp[i \arg(1 + \lambda)] = \sqrt{[(1 + \alpha)^2 + \beta^2]} \exp(i\epsilon)$$

where $\tan \epsilon = \beta/(1 + \alpha)$ and $(r/a)^\lambda = (r/a)^\alpha \exp[i\beta \ln(r/a)]$, hence for $r < a$

$$\psi(r, \theta) = \frac{2B_0 a}{\sqrt{(1 + \alpha)^2 + \beta^2}} \left(\frac{r}{a}\right)^\alpha \exp[i(\beta \ln(r/a) + \theta - \epsilon)].$$

The specified form follows on taking the imaginary part.

A2) Here it is convenient to return to the more fundamental equations. With velocity field $\mathbf{v} = v(z)\hat{\mathbf{i}}$ and applied magnetic field $B_0\hat{\mathbf{k}}$, Ohm's law (2.92) renders the current flow density $j_y = \sigma(E_0 - B_0 v)$. From (2.56) and (3.4) we have $\nabla p = \mathbf{j} \times \mathbf{B} + \mu \nabla^2 \mathbf{v}$ for the steady unidirectional flow, with x -component

$$-G = B_0 j_y + \mu \frac{d^2 v}{dz^2} = B_0 \sigma (E_0 - B_0 v) + \mu \frac{d^2 v}{dz^2}$$

such that on writing $M = B_0 a \sqrt{\sigma/\mu}$ (the Hartmann number) we have

$$\frac{d^2 v}{dz^2} - \frac{M^2}{a^2} v = -\frac{1}{\mu} (G + \sigma E_0 B_0) = 0,$$

which is the required differential equation for the velocity profile. Now the general solution of this differential equation is

$$v(z) = c_1 \cosh \frac{Mz}{a} + c_2 \sinh \frac{Mz}{a} + \frac{a^2}{M^2 \mu} (G + \sigma E_0 B_0);$$

and since $v(\pm a) = 0$ on the rigid boundary, the solution for the velocity profile is

$$v(z) = \frac{a^2}{M^2 \mu} (G + \sigma E_0 B_0) \left(1 - \frac{\cosh(Mz/a)}{\cosh M} \right).$$

For insulated walls

$$\int_{-a}^a j_y dz = \int_{-a}^a \sigma(E_0 - B_0 v) dz = 0 \implies E_0 = \frac{B_0}{2a} \int_{-a}^a v(z) dz;$$

and since

$$\int_{-a}^a \left(1 - \frac{\cosh(Mz/a)}{\cosh M} \right) dz = 2a \left(1 - \frac{\tanh M}{M} \right)$$

we obtain

$$E_0 = \frac{B_0 a^2}{M^2 \mu} (G + \sigma E_0 B_0) \left(1 - \frac{\tanh M}{M} \right),$$

which we can rearrange to extract

$$E_0 = \frac{B_0 a^2 G}{M^2 \mu} \left(\frac{M}{\tanh M} - 1 \right) = \frac{G}{\sigma B_0} \left(\frac{M}{\tanh M} - 1 \right)$$

on recalling that $M = B_0 a \sqrt{\sigma/\mu}$. Thus for $M \gg 1$ we have $E_0 \simeq GM/(\sigma B_0)$, such that

$$v(0) \simeq \frac{a^2}{M^2 \mu} (G + \sigma E_0 B_0) \simeq \frac{a^2}{M^2 \mu} (G + GM) \simeq \frac{U_0}{M}$$

where $U_0 = Ga^2/\mu$. On the other hand, for a conducting duct we have $v(0) \simeq U_0/M^2$ when $M \gg 1$, given that $E_0 = 0$ in that case.

Section 5.7

A1) In Cartesian coordinates, on adopting $\mathbf{v} = v(x, y) \hat{\mathbf{i}}$ and $\mathbf{B} = B \hat{\mathbf{k}}$ we have³

$$\mathbf{v} \times \mathbf{B} = -vB \hat{\mathbf{j}} \implies \nabla \times (\mathbf{v} \times \mathbf{B}) = -\hat{\mathbf{k}} \frac{\partial}{\partial x} (vB) = -\hat{\mathbf{k}} \nabla \cdot (B\mathbf{v}),$$

so the nontrivial z -component of the ideal equation of magnetic induction (5.38) is

$$0 = \frac{\partial B}{\partial t} + \nabla \cdot (B\mathbf{v}) = \frac{\partial B}{\partial t} + \mathbf{v} \cdot \nabla B + B \nabla \cdot \mathbf{v} = \frac{dB}{dt} - \frac{B}{\rho} \frac{d\rho}{dt} = \rho \frac{d}{dt} \left(\frac{B}{\rho} \right)$$

³In the magnetic plasma confinement literature, for Cartesian “slab” coordinates it is usually assumed that any equilibrium density or pressure variation is in the “radial” x -direction, with any “poloidal” component in the y -direction and the “toroidal” magnetic field in the z -direction. On the other hand, in the plasma astrophysics literature the density or pressure variation is often assumed to be in the z -direction and the magnetic field in the xy -plane, so there is a cyclic permutation of the Cartesian coordinates [Glossary Section “Slab Model”].

on invoking (5.33), so we have the first result that the ratio B/ρ is conserved. Now (5.39) may be rewritten in a form analogous to (3.20) — viz.

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{1}{2} v^2 + h + V + \frac{B^2}{2\mu_0} \right) = \mathbf{v} \times (\nabla \times \mathbf{v}),$$

since the magnetic stress tensor $\mathcal{T} = B^2/(2\mu_0) (\mathbf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}) = B^2/(2\mu_0) \mathbf{I}_\perp$ is such that $\nabla \cdot \mathcal{T} = \nabla B^2/(2\mu_0)$ provided $\mathbf{B} = B(x, y) \hat{\mathbf{k}}$ so that $\hat{\mathbf{k}} \cdot \nabla B^2/(2\mu_0) = 0$. Thus for the steady unidirectional (irrotational) flow field $\mathbf{v} = v(x, y) \hat{\mathbf{i}}$, we have the modified Bernoulli equation

$$\frac{1}{2} v^2 + h + V + \frac{B^2}{2\mu_0} = \text{const}$$

throughout the flow, where we may write $\mathbf{v} = -\nabla \phi$ as in Chapter 3.

Section 5.9

A1) In a magnetohydrostatic configuration, the dispersion relation obtained from (5.59) may be written (dropping the prime on ignoring background flow)

$$(\rho \omega^2 - \mu_0^{-1} (\mathbf{k} \cdot \mathbf{B})^2) [(\rho \omega^2)^2 - (\gamma p + \mu_0^{-1} B^2) k^2 \rho \omega^2 + \gamma p k^2 (\mathbf{k} \cdot \mathbf{B})^2] = 0.$$

The first bracket of course corresponds to the Alfvén wave, and the second bracket (a quadratic in $\rho \omega^2$) has roots

$$\rho \omega^2 = \frac{1}{2} \left[(\gamma p + \mu_0^{-1} B^2) \left(k^2 \pm \sqrt{1 - \frac{4\gamma p k^2 (\mathbf{k} \cdot \mathbf{B})^2}{(\gamma p + \mu_0^{-1} B^2)^2}} \right) \right],$$

whence the dispersion relations for the slow and fast magnetosonic waves corresponding to the negative and positive signs respectively, since

$$4\gamma p k^2 (\mathbf{k} \cdot \mathbf{B})^2 / (\gamma p + \mu_0^{-1} B^2)^2 \ll 1$$

when $k_\parallel \ll k_\perp$. The given forms for the corresponding Lagrangian displacements can be verified to be nontrivial solutions such that $\mathbf{D} \cdot \boldsymbol{\xi} = 0$, with $\mathbf{D}$ given by (5.55). (The more mathematically inclined reader may also care to ascertain the uniqueness of these forms.)

Section 5.10

A1) $\nabla \psi(x, y) = \hat{\mathbf{i}} \partial \psi / \partial x + \hat{\mathbf{j}} \partial \psi / \partial y$ and $\nabla z = \hat{\mathbf{k}}$, so $\mathbf{B} = \nabla z \times \nabla \psi + F(\psi) \nabla z$ is $\mathbf{B} = \hat{\mathbf{k}} \times (\hat{\mathbf{i}} \partial \psi / \partial x + \hat{\mathbf{j}} \partial \psi / \partial y) + F(\psi) \hat{\mathbf{k}} = (-\partial \psi / \partial y) \hat{\mathbf{i}} + \partial \psi / \partial x \hat{\mathbf{j}} + F(\psi) \hat{\mathbf{k}}$. Hence $\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \hat{\mathbf{i}} F'(\psi) \partial \psi / \partial y - \hat{\mathbf{j}} F'(\psi) \partial \psi / \partial x + \hat{\mathbf{k}} \nabla^2 \psi$ such that $\mu_0 \nabla p = \mu_0 \mathbf{j} \times \mathbf{B} = -F(\psi) F'(\psi) \nabla \psi - \nabla^2 \psi \nabla \psi$, and the Grad-Shafranov analogue follows on rearranging, after noting that $\nabla p(\psi) = p'(\psi) \nabla \psi$.

$$\text{A2)} \mathbf{j} = \mu_0^{-1} \nabla \times \mathbf{B} = \mu_0^{-1} \alpha(\mathbf{r}) \mathbf{B} \implies \mathbf{j} \cdot \nabla \alpha = \mu_0^{-1} \alpha(\mathbf{r}) \mathbf{B} \cdot \nabla \alpha = 0.$$

A3) Vary the equilibrium by making infinitesimal displacements $\delta \mathbf{r}(\mathbf{r}_0, t)$ of each fluid element away from its rest position $\mathbf{r}_0$, where the time variation is arbitrary. Then the corresponding Eulerian variation in the velocity field is $\delta \mathbf{v} = \partial(\delta \mathbf{r})/\partial t$ — cf. Sect.2.4 for the Eulerian description. Now vary both sides of the adiabatic equation of state (5.35) and integrate with respect to time to get $\delta p = -\gamma p \nabla \cdot \delta \mathbf{r} - \delta \mathbf{r} \cdot \nabla p$. Also vary the ideal magnetic induction equation (5.38) to get $\delta \mathbf{B} = \nabla \times (\delta \mathbf{r} \times \mathbf{B})$. Then varying (5.70) we obtain

$$\begin{aligned} \delta W &= \int_{\mathcal{P}} \left(\frac{-\gamma p \nabla \cdot \delta \mathbf{r} - \delta \mathbf{r} \cdot \nabla p}{\gamma - 1} + \frac{\mathbf{B} \cdot \nabla \times (\delta \mathbf{r} \times \mathbf{B})}{\mu_0} \right) d\tau \\ &= \int_{\mathcal{P}} \left(\nabla p + \frac{\mathbf{B} \times (\nabla \times \mathbf{B})}{\mu_0} \right) \cdot \delta \mathbf{r} d\tau, \end{aligned}$$

on integrating by parts and neglecting boundary terms. For all variations $\delta \mathbf{r}$ not involving the boundary $\partial \mathcal{P}$, we therefore have $\delta W = 0$ if and only if $\nabla p = \mu_0^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}$.

The second part of the question is much easier, as neither the magnetic field nor the pressure is required to vary with the fluid in Taylor relaxation. Indeed, the pressure is constant assuming the boundaries are fixed, and the magnetic field is constrained only by the divergence-free condition ensured by the vector potential representation, giving $\delta \mathbf{B} = \nabla \times \delta \mathbf{A}$. Thus from (5.70) and (5.71), the varied (5.72) gives

$$\begin{aligned} \delta W &= \int_{\mathcal{P}} \frac{2\mathbf{B} \cdot (\nabla \times \delta \mathbf{A}) - \lambda(\mathbf{A} \cdot \nabla \times \delta \mathbf{A} + \delta \mathbf{A} \cdot \mathbf{B})}{2\mu_0} d\tau \\ &= \int_{\mathcal{P}} \frac{\nabla \times \mathbf{B} - \lambda \mathbf{B}}{\mu_0} \cdot \delta \mathbf{A} d\tau \end{aligned}$$

on using (1.39) and the Divergence Theorem (1.60), and neglecting the boundary terms. Consequently, $\delta W = 0$ if and only if $\nabla \times \mathbf{B} = \lambda \mathbf{B}$.

Section 5.11

A1) From (1.9) and (1.41) we first rewrite (5.78) as

$$\mathbf{B} = J^{-1} \psi'(s) [\mathbf{e}_\theta + q(s) \mathbf{e}_\zeta],$$

where J is the Jacobian defined after (5.74). From (1.3–1.6) and (1.54) the vector area element on $S_P: \theta = \text{const}$ is $d\mathbf{S}_\theta = \mathbf{e}_\zeta \times \mathbf{e}_s d\zeta ds = J \mathbf{e}^\theta d\zeta ds$ and that on $S_T: \zeta = \text{const}$ is $d\mathbf{S}_\zeta = \mathbf{e}_s \times \mathbf{e}_\theta ds d\theta = J \mathbf{e}^\zeta ds d\theta$. Thus using (1.7), $\mathbf{B} \cdot d\mathbf{S}_\theta = \psi'(s) d\zeta ds$ and $\mathbf{B} \cdot d\mathbf{S}_\zeta = q(s) \psi'(s) ds d\theta$ irrespective of the constants or particular straight-field-line coordinate system chosen. Hence

$$\Psi_P = \int_0^{2\pi} d\zeta \int_0^s \psi'(s') ds' = \int_0^s 2\pi \psi'(s') ds' = 2\pi [\psi(s) - \psi(0)]$$

in agreement with (5.75), and

$$\Psi_T = \int_0^{2\pi} d\theta \int_0^s q(s') \psi'(s') ds' = \int_0^s 2\pi q(s') \psi'(s') ds' ,$$

which is in the same form as (5.76) with the identification

$$q(s) = \frac{F(\psi)}{2\pi} \oint_\theta \frac{dl}{R|\nabla\psi|} , \quad \nabla\psi(s) = \psi'(s) \nabla s .$$

Section 5.12

A1) For static MHD equilibria, we have the simplification that $\mathbf{v} = 0$. We are told that B_z is discontinuous, so the case (3)(b) “tangential discontinuities” in the summary at the end of Sect.5.12 is relevant. Our task is to translate the tangential boundary condition $\hat{\mathbf{n}} \cdot \mathbf{B}_\pm = 0$ and the pressure balance jump condition (5.103) into the representation $\mathbf{B} = \nabla z \times \nabla\psi + F(\psi)\nabla z$, $\psi = \psi(x, y)$.

First observe that ψ must be continuous across S , to avoid the physical impossibility of an infinite magnetic field from a δ -function in $\nabla\psi$. The jump condition is

$$\left[\left[p + \frac{|\nabla\psi|^2 + F^2}{2\mu_0} \right] \right] = 0 .$$

The tangential boundary condition is $\hat{\mathbf{n}} \cdot (\nabla z \times \nabla\psi_\pm + F\nabla z) = 0$, where the second term vanishes immediately because $\hat{\mathbf{n}}$ is in the (x, y) -plane. The first term can vanish only if $\hat{\mathbf{n}} \times \nabla\psi_\pm = 0$, so S and the surface $\psi_\pm = \text{const}$ must coincide.

A2) In the complex variable method, the discontinuity arises from a branch cut in the complex plane. Without loss of generality, we can use units such that $\Delta = B_0 = 1$. The cut is then on the imaginary axis between $y = -1$ and $y = +1$, so $1 - y^2 > 0$ on the current sheet. Thus $\sqrt{1 - y^2}$ is real and $|iy + \sqrt{1 - y^2}| = 1$, which will be useful when we use the identity $\ln z = \ln |z| + i \arg z$ below.

From (Q1) we need to show *continuity* of $\psi = \Re w(\zeta)$ and $|\nabla\psi| = |\Re w'(\zeta)|$ across the cut, and also *constancy* of $\psi = \Re w(\zeta)$ along the cut. In the neighbourhood of the cut, x is infinitesimal and only its sign is important, which we can represent by writing $\zeta = \pm 0 + iy$. Thus

$$\begin{aligned} \Re w(\pm 0 + iy) &= \Re \left\{ \pm \left[iy\sqrt{1 - y^2} + \ln \left(iy + \sqrt{1 - y^2} \right) \right] \right\} \\ &= \Re \left\{ \pm \left[iy\sqrt{1 - y^2} + i \arcsin y \right] \right\} \equiv 0 , \end{aligned}$$

which demonstrates the continuity and constancy of ψ . Now we observe that

$$\Re w'(\zeta) = 2 \operatorname{sgn}(\Re \zeta) \Re \sqrt{1 + \zeta^2} ,$$

which is discontinuous across the cut as expected. However, its absolute value $|\Re w'(\zeta)|$ and hence $|\nabla \psi|^2$ are continuous, so the pressure balance jump condition is satisfied.

A3) Let $v \equiv \hat{\mathbf{n}} \cdot \mathbf{v}$ be the component of the shock-frame fluid velocity in the direction of $\hat{\mathbf{n}}$, the unit normal to the shock front, the shock frame being the Galilean frame in which the shock front is stationary and $\mathbf{v} = v\hat{\mathbf{n}}$. The derivation of the jump condition $[\![\rho v]\!] = 0$ is the same as in Sect.4.11 for ordinary fluids and does not need to be repeated here.

Neglecting gravity and recalling that in the shock frame we can drop the $\partial/\partial t$ terms in the conservation equations, on dotting both sides of the MHD momentum conservation equation (5.41) with $\hat{\mathbf{n}}$ and integrating for an infinitesimal distance across the shock (treated as a sharp discontinuity), we obtain the continuity condition $\hat{\mathbf{n}} \cdot [\![\rho \mathbf{v} \mathbf{v} + p \mathbf{I} + \mathcal{T}]\!] \cdot \hat{\mathbf{n}} = 0$. Since we are told $\mathbf{B}$ is normal to $\hat{\mathbf{n}}$, from (5.40) we have $\hat{\mathbf{n}} \cdot \mathcal{T} \cdot \hat{\mathbf{n}} = B^2/(2\mu_0)$. Thus we obtain

$$[\![\rho v^2 + p + B^2/2\mu_0]\!] = 0 ,$$

the generalisation of (4.128) that involves the unknowns v_2^2 and v_1^2 where the subscripts 1 and 2 distinguish the two sides of the shock. The squared mass continuity condition $[\![\rho^2 v^2]\!] = 0$ provides an independent equation involving v_2^2 and v_1^2 , allowing us to rewrite this second generalised Rankine-Hugoniot equation as

$$\left[\left[\frac{v^2}{2} \right] \right] = -\overline{1/\rho} \left[\left[p + \frac{B^2}{2\mu_0} \right] \right] ,$$

where the bar denotes the average quantity — i.e. $\overline{1/\rho} = (1/\rho_1 + 1/\rho_2)/2$. Likewise, from the energy equation (5.45) and $\hat{\mathbf{n}} \cdot \mathcal{T} \cdot \hat{\mathbf{n}} = B^2/2\mu_0$, on invoking the continuity of $\rho \mathbf{v}$ we have the third equation generalising (4.131) — viz.

$$\left[\left[\frac{1}{2} v^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{B^2}{\mu_0 \rho} \right] \right] = 0 ,$$

which we rewrite as

$$\left[\left[\frac{v^2}{2} + \frac{1}{\rho} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) + \frac{1}{\rho} \left(p + \frac{B^2}{2\mu_0} \right) \right] \right] = 0 .$$

Eliminating the $v^2/2$ component between these second and third generalised equations produces

$$\begin{aligned} 0 &= \frac{1}{\gamma - 1} \left[\left[\frac{p}{\rho} \right] \right] + \left[\left[\frac{B^2}{2\mu_0 \rho} \right] \right] + \left[\left[\frac{1}{\rho} \left(p + \frac{B^2}{2\mu_0} \right) \right] \right] - \overline{1/\rho} \left[\left[p + \frac{B^2}{2\mu_0} \right] \right] \\ &= \frac{1}{\gamma - 1} \left[\left[\frac{p}{\rho} \right] \right] + \left[\left[\frac{B^2}{2\mu_0 \rho} \right] \right] + \left[\left[\frac{1}{\rho} \right] \right] \overline{\left(p + \frac{B^2}{2\mu_0} \right)} . \end{aligned}$$

Section 5.15

A1) (a) Recalling the plane wave form (4.2), for $\mathbf{k} = k\hat{\mathbf{j}}$ (parallel propagation) the given perturbation equation becomes

$$\omega \mathbf{B}_1 = \frac{kB}{\mu_0 n e} \nabla \times \mathbf{B}_1$$

where $\nabla \times \mathbf{B}_1 = ikB_{1z}\hat{\mathbf{i}} - ikB_{1x}\hat{\mathbf{k}}$, such that $B_{1y} = 0$ and

$$\begin{aligned} \omega B_{1x} &= i \frac{k^2 B}{\mu_0 n e} B_{1z}, \\ \omega B_{1z} &= -i \frac{k^2 B}{\mu_0 n e} B_{1x}. \end{aligned}$$

These two simultaneous equations have a nontrivial solution $(B_{1x}, B_{1z}) \neq \mathbf{0}$ when

$$\omega^2 - \left(\frac{k^2 B}{\mu_0 n e} \right)^2 = 0 \quad \implies \quad \omega = \frac{k^2 B}{\mu_0 n e}.$$

(b) With $\nabla n(z) = (dn/dz)\hat{\mathbf{k}}$, for the plane wave form (4.2) when $\mathbf{k} = k\hat{\mathbf{i}}$ (transverse propagation) the given perturbation equation becomes

$$-i\omega \mathbf{B}_1 = \frac{1}{\mu_0 n^2 e} [(\nabla \times \mathbf{B}_1) \cdot \nabla n] \mathbf{B}$$

where $\nabla \times \mathbf{B}_1 = -ikB_{1z}\hat{\mathbf{j}} + ikB_{1y}\hat{\mathbf{k}}$, such that $(B_{1x}, B_{1z}) = \mathbf{0}$ and

$$\omega B_{1y} = \frac{kB}{\mu_0 n e} \frac{1}{n} \frac{dn}{dz} B_{1y} \quad \implies \quad \omega = \frac{kB}{\mu_0 n e} \kappa, \text{ where } \kappa \equiv \frac{1}{n} \frac{dn}{dz}.$$

Case (a) describes simple whistler waves and case (b) simple Hall drift waves, with one case the mathematical “counterpoint” of the other. The expression for the frequency of the whistler wave may be rewritten $\omega\omega_{ci} = k^2 c_A^2$, where $\omega_{ci} = eB/m_i$ is the ion gyrofrequency and c_A is the Alfvén wave speed as before, so at higher frequencies ($\omega > \omega_{ci}$) the whistler wave phase speed ω/k exceeds the Alfvén speed — i.e. high-frequency whistler waves propagate along the magnetic field faster than the Alfvén waves of ideal MHD discussed in Sect.5.9, where not only the electrons but also the ions are considered to be “frozen-in”. The expression for the frequency of the Hall drift wave propagating in the $\mathbf{B} \times \nabla n$ direction may be rewritten $\omega\omega_{ci} = kc_A^2 \kappa$, so at similar higher frequencies the Hall drift phase speed ω/k may likewise exceed the Alfvén speed c_A unless the wave number is small ($k < \kappa$).

Section 6.4

A1) The form $v_{1z} = C \exp(-kz)$ appropriately satisfies the boundary condition $v_{1z} \rightarrow 0$ as $z \rightarrow \infty$, and substitution into the governing differential equation immediately produces $1 + gk/\omega^2 = 0$. Thus we again obtain the classical dispersion relation $\omega^2 = -gk$, giving the maximum Rayleigh-Taylor growth rate $\sqrt{gk}$ identified in Sect. 4.7 for these modes localised near $z = 0$ — and likewise when $\rho_1 \gg \rho_2$ for the interchange modes localised near the density discontinuity mentioned in Sect. 6.3, with $k_{\parallel} = 0$ everywhere in the assumed uniform magnetic field. The localisation at the free surface or interface, and the fast growth rate independent of the density profile, are the chief characteristics of this so-called “global” Rayleigh-Taylor instability. The corresponding Rayleigh variational form

$$\omega^2 = - \frac{gk^2 \int (d\rho/dz) |v_{1z}|^2 dz}{\int \rho (|dv_{1z}/dz|^2 + k^2 |v_{1z}|^2) dz} ,$$

obtained from the governing differential equation by integration over the entire plasma domain as before (on assuming the plasma to be confined by horizontal rigid walls or infinitely deep such that v_{1z} vanishes at the plasma boundaries), of course does involve the density profile $\rho(z)$. However, when $\rho(z) \simeq \rho_1 H(z)$ such that $d\rho/dz \simeq \rho_1 \delta(z)$ where $\delta(z)$ is the Dirac delta function, the maximum growth rate $\sqrt{gk}$ of the “global” Rayleigh-Taylor modes corresponding to $v_{1z} = C \exp(-kz)$ in the half-space $z > 0$ is immediately recovered.

Section 6.5

A1) In the derivation, equilibrium properties (e.g. $\mathbf{B} \cdot \nabla p = 0$) and boundary conditions (e.g. $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$) are occasionally invoked. From the identities

$$\begin{aligned} \boldsymbol{\xi}^* \cdot (\nabla \times \mathbf{B}_1) \times \mathbf{B} &= \nabla \cdot [(\boldsymbol{\xi}^* \times \mathbf{B}) \times \mathbf{B}_1] - \mathbf{B}_1 \cdot \nabla \times (\boldsymbol{\xi}^* \times \mathbf{B}) , \\ \boldsymbol{\xi}^* \cdot \nabla (\gamma p \nabla \cdot \boldsymbol{\xi}) &= \nabla \cdot (\gamma p \nabla \cdot \boldsymbol{\xi} \boldsymbol{\xi}^*) - \gamma p |\nabla \cdot \boldsymbol{\xi}|^2 \end{aligned}$$

and neglecting the gravity term, equation (6.41) may be re-expressed as

$$\begin{aligned} \Delta W[\boldsymbol{\xi}, \boldsymbol{\xi}] &= \frac{1}{2} \int [\mu_0^{-1} |\mathbf{B}_1|^2 - \boldsymbol{\xi}^* \cdot [\mathbf{j} \times \mathbf{B}_1 + \nabla (\boldsymbol{\xi} \cdot \nabla p)] + \gamma p |\nabla \cdot \boldsymbol{\xi}|^2] d\tau \\ &\quad + \frac{1}{2} \int (\mu_0^{-1} \mathbf{B} \cdot \mathbf{B}_1 - \gamma p \nabla \cdot \boldsymbol{\xi}) \boldsymbol{\xi}^* \cdot d\mathbf{S} , \end{aligned}$$

invoking the Divergence Theorem (1.60) and noting that $\mathbf{B}_1^* = \nabla \times (\boldsymbol{\xi}^* \times \mathbf{B})$. Since we now assume $\nabla p = \mathbf{j} \times \mathbf{B}$ (and have $\nabla \times \nabla p = 0$ and $\nabla \cdot \mathbf{B} = 0$),

$$\mathbf{B} \cdot \mathbf{j} \times \mathbf{B}_1 = -\mathbf{B}_1 \cdot \mathbf{j} \times \mathbf{B} = -\mathbf{B}_1 \cdot \nabla p = \nabla \cdot [(\nabla p) \times (\boldsymbol{\xi} \times \mathbf{B})] = -\nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p) \mathbf{B}]$$

and

$$\mathbf{B} \cdot \nabla (\boldsymbol{\xi} \cdot \nabla p) = \nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p) \mathbf{B}] .$$

Thus the parallel components of $\mathbf{j} \times \mathbf{B}_1$ and $\nabla (\boldsymbol{\xi} \cdot \nabla p)$ cancel, so the nonzero contribution from the corresponding dot product in the volume integral only involves the *perpendicular* component of $\boldsymbol{\xi}^*$. Hence on writing $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \hat{\mathbf{b}}$ (with $\hat{\mathbf{b}} = \mathbf{B}/B$) and invoking the identity

$$\boldsymbol{\xi}_\perp^* \cdot \nabla (\boldsymbol{\xi} \cdot \nabla p) = \nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p) \boldsymbol{\xi}_\perp^*] - (\boldsymbol{\xi} \cdot \nabla p) \nabla \cdot \boldsymbol{\xi}_\perp^* ,$$

since $\boldsymbol{\xi} \cdot \nabla p = \boldsymbol{\xi}_\perp \cdot \nabla p$ we obtain

$$\Delta W[\boldsymbol{\xi}, \boldsymbol{\xi}] = \Delta W_p + \frac{1}{2} \int [\mu_0^{-1} \mathbf{B} \cdot \mathbf{B}_1 - \boldsymbol{\xi}_\perp \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}] \boldsymbol{\xi}_\perp^* \cdot d\mathbf{S}$$

where the plasma volume integral is reduced to

$$\Delta W_p = \frac{1}{2} \int [\mu_0^{-1} |\mathbf{B}_1|^2 - \boldsymbol{\xi}_\perp^* \cdot \mathbf{j} \times \mathbf{B}_1 + (\nabla \cdot \boldsymbol{\xi}_\perp^*) \boldsymbol{\xi}_\perp \cdot \nabla p + \gamma p |\nabla \cdot \boldsymbol{\xi}|^2] d\tau .$$

The non-cavitation kinematic condition is again adopted with $\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 + \hat{\mathbf{n}}_1$ at any point $\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\xi}$ on the perturbed interface, where $\hat{\mathbf{n}}_0$ denotes the unit normal at the point $\mathbf{r}_0$ on the unperturbed interface and $\hat{\mathbf{n}}_1$ its increment under any small plasma displacement $\boldsymbol{\xi}$. On noting the relevant first order Taylor expansions under a displacement $\boldsymbol{\xi}_\perp$, from (5.95) the continuity of total pressure boundary condition at the perturbed interface is

$$p_1 + \frac{\mathbf{B} \cdot \mathbf{B}_1}{\mu_0} + \boldsymbol{\xi}_\perp \cdot \nabla \left(p + \frac{B^2}{2\mu_0} \right) = \frac{\mathbf{B}^v \cdot \mathbf{B}_1^v}{\mu_0} + \boldsymbol{\xi}_\perp \cdot \nabla \frac{(B^v)^2}{2\mu_0}$$

where the superscript v denotes the vacuum quantities — but this condition may then be considered to apply at the unperturbed interface, as were the free surface conditions (4.49) and (4.51) in the linearised theory of water waves. From (6.27), this boundary condition may be rewritten as

$$-\gamma p \nabla \cdot \boldsymbol{\xi} + \mu_0^{-1} \mathbf{B} \cdot \mathbf{B}_1 + \boldsymbol{\xi}_\perp \cdot \nabla \frac{B^2}{2\mu_0} = \mu_0^{-1} \mathbf{B}^v \cdot \mathbf{B}_1^v + \boldsymbol{\xi}_\perp \cdot \nabla \frac{(B^v)^2}{2\mu_0}$$

and invoked in the surface integral, such that

$$\begin{aligned} & \frac{1}{2} \int [\mu_0^{-1} \mathbf{B} \cdot \mathbf{B}_1 - \boldsymbol{\xi}_\perp \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}] \boldsymbol{\xi}_\perp^* \cdot d\mathbf{S} \\ &= \frac{1}{2} \int \left[\boldsymbol{\xi}_\perp \cdot \nabla \left(p + \frac{B^2}{2\mu_0} \right) - \boldsymbol{\xi}_\perp \cdot \nabla \frac{(B^v)^2}{2\mu_0} - \mu_0^{-1} \mathbf{B}^v \cdot \mathbf{B}_1^v \right] \boldsymbol{\xi}_\perp^* \cdot d\mathbf{S} \\ &= \frac{1}{2} \int |\hat{\mathbf{n}} \cdot \boldsymbol{\xi}_\perp|^2 \left[\nabla \left(p + \frac{B^2}{2\mu_0} \right) \right] \cdot d\mathbf{S} - \frac{1}{2\mu_0} \int \mathbf{B}^v \cdot \mathbf{B}_1^v \boldsymbol{\xi}_\perp^* \cdot d\mathbf{S} , \end{aligned}$$

since $\hat{\mathbf{n}} \times \nabla [p + B^2/(2\mu_0)]$ is continuous across the boundary.

Recall from Sect.5.13 that a velocity field $\mathbf{v}$ may be defined in the vacuum, to within an arbitrary parallel component $\hat{\mathbf{b}}_v \cdot \mathbf{v}$. Invoking the plasma equation $\mathbf{E}_1 + \mathbf{v}_{1\perp} \times \mathbf{B} = 0$, the condition $\hat{\mathbf{n}} \times \mathbf{E}_1^v + \hat{\mathbf{n}} \times (\mathbf{v}_{1\perp} \times \mathbf{B}) = \hat{\mathbf{n}} \cdot \mathbf{v}_{1\perp} (\mathbf{B}^v - \mathbf{B})$ from (5.101) at the perturbed boundary reduces to $\hat{\mathbf{n}} \times \mathbf{E}_1^v = \hat{\mathbf{n}} \cdot \mathbf{v}_{1\perp} \mathbf{B}^v$. If the

perturbed vector potential $\mathbf{A}_1$ is now introduced such that $\mathbf{B}_1^v = \nabla \times \mathbf{A}_1$, by integrating with respect to time t we therefore have $-\hat{\mathbf{n}} \times \mathbf{A}_1 = \hat{\mathbf{n}} \cdot \boldsymbol{\xi}_\perp \mathbf{B}^v$, on adopting the Coulomb gauge where $\mathbf{E}_1^v = -\partial_t \mathbf{A}_1$. Consequently,

$$\begin{aligned} -\frac{1}{2\mu_0} \int \mathbf{B}^v \cdot \mathbf{B}_1^v \boldsymbol{\xi}_\perp^* \cdot d\mathbf{S} &= \frac{1}{2\mu_0} \int (\nabla \times \mathbf{A}_1) \cdot (\hat{\mathbf{n}} \times \mathbf{A}_1^*) d\mathbf{S} \\ &= \frac{1}{2\mu_0} \int d\mathbf{S} \cdot [\mathbf{A}_1^* \times (\nabla \times \mathbf{A}_1)] \\ &= \int |\mathbf{B}_1^v|^2 / (2\mu_0) d\tau^v, \end{aligned}$$

since $\nabla \times \mathbf{B}_1^v = \nabla \times \nabla \times \mathbf{A}_1 = 0$ because there is no current flow in the vacuum region. In applying the Divergence Theorem (1.60) over this vacuum region, we note that it is implicit that there is no contribution to the surface integral over the outer boundary of the physical system, which is justifiable if this boundary is assumed to be perfectly conducting or at infinity.

A2) Since $\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B})$,

$$\begin{aligned} \mathbf{B}_1 \cdot \mathbf{B} &= \mathbf{B} \cdot \mathbf{B}_1 = \mathbf{B} \cdot (\mathbf{B} \cdot \nabla \boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp \cdot \nabla \mathbf{B} - \mathbf{B} \nabla \cdot \boldsymbol{\xi}_\perp) \\ &= \mathbf{B} \cdot \nabla (\boldsymbol{\xi}_\perp \cdot \mathbf{B}) - \boldsymbol{\xi}_\perp \cdot (\mathbf{B} \cdot \nabla \mathbf{B}) \\ &\quad - \boldsymbol{\xi}_\perp \cdot \nabla (B^2/2) - B^2 \nabla \cdot \boldsymbol{\xi}_\perp \\ &= -2B^2 \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} + \mu_0 \boldsymbol{\xi}_\perp \cdot \nabla p - B^2 \nabla \cdot \boldsymbol{\xi}_\perp, \end{aligned}$$

because $\boldsymbol{\xi}_\perp \cdot \mathbf{B} = 0$, $\nabla (B^2/2) = \mathbf{B} \cdot \nabla \mathbf{B} - \mu_0 \nabla p$ from $\nabla p = \mu_0^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B}$, and $\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa} = B^{-2} \boldsymbol{\xi}_\perp \cdot (\mathbf{B} \cdot \nabla \mathbf{B})$ follows on dotting (5.31) with $\boldsymbol{\xi}_\perp$. After rearranging and taking the complex conjugate to render $\nabla \cdot \boldsymbol{\xi}_\perp^*$ from this result, on multiplying through by $\boldsymbol{\xi}_\perp \cdot \nabla p$ we obtain

$$(\nabla \cdot \boldsymbol{\xi}_\perp^*) \boldsymbol{\xi}_\perp \cdot \nabla p = -B^{-2} \mathbf{B}_1^* \cdot \mathbf{B} \boldsymbol{\xi}_\perp \cdot \nabla p + \mu_0 B^{-2} |\boldsymbol{\xi}_\perp \cdot \nabla p|^2 - 2\boldsymbol{\xi}_\perp^* \cdot \boldsymbol{\kappa} \boldsymbol{\xi}_\perp \cdot \nabla p.$$

Writing $\mathbf{j} = \sigma \mathbf{B} + B^{-2} \mathbf{B} \times \nabla p$ where $\sigma = B^{-2} \mathbf{j} \cdot \mathbf{B}$ (given $\nabla p = \mathbf{j} \times \mathbf{B}$), we also have

$$\boldsymbol{\xi}_\perp^* \cdot \mathbf{j} \times \mathbf{B}_1 = B^{-2} \mathbf{B}_1 \cdot \mathbf{B} \boldsymbol{\xi}_\perp^* \cdot \nabla p + \sigma \boldsymbol{\xi}_\perp^* \times \mathbf{B} \cdot \mathbf{B}_1.$$

Since also $|\mathbf{B}_1|^2 = |\mathbf{B}_{1\perp}|^2 + B^{-2} |\mathbf{B} \cdot \mathbf{B}_1|^2$, we consequently obtain the form (6.49) from (6.46), on re-expressing the combination

$$\mu_0^{-1} |\mathbf{B}_1|^2 - \boldsymbol{\xi}_\perp^* \cdot \mathbf{j} \times \mathbf{B}_1 + (\nabla \cdot \boldsymbol{\xi}_\perp^*) \boldsymbol{\xi}_\perp \cdot \nabla p$$

and considering the expansion of the right-hand side of

$$|\mathbf{B}_1 \cdot \mathbf{B} - \mu_0 \boldsymbol{\xi}_\perp \cdot \nabla p|^2 = (\mathbf{B}_1 \cdot \mathbf{B} - \mu_0 \boldsymbol{\xi}_\perp \cdot \nabla p)(\mathbf{B}_1^* \cdot \mathbf{B} - \mu_0 \boldsymbol{\xi}_\perp^* \cdot \nabla p).$$

Section 6.6

A1) $\boldsymbol{\xi} \times \mathbf{B} = (\xi_\theta B_z - \xi_z B_\theta) \hat{\mathbf{e}}_r - \xi_r B_z \hat{\mathbf{e}}_\theta + \xi_r B_\theta \hat{\mathbf{e}}_z = -i\zeta \hat{\mathbf{e}}_r - \xi B_z \hat{\mathbf{e}}_\theta + \xi B_\theta \hat{\mathbf{e}}_z$,
hence

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$$

$$\begin{aligned} &= i \left(\frac{m}{r} \xi B_\theta + k_z \xi B_z \right) \hat{\mathbf{e}}_r + \left(i k_z (-i\zeta) - \frac{d}{dr} (\xi B_\theta) \right) \hat{\mathbf{e}}_\theta \\ &\quad + \left(\frac{1}{r} \frac{d}{dr} (-\xi B_z) - \frac{im}{r} (-i\zeta) \right) \hat{\mathbf{e}}_z \\ &= i \left(\frac{m}{r} B_\theta + k_z B_z \right) \xi \hat{\mathbf{e}}_r + \left(k_z \zeta - \frac{d(\xi B_\theta)}{dr} \right) \hat{\mathbf{e}}_\theta - \left(\frac{1}{r} \frac{d(r\xi B_z)}{dr} + \frac{m}{r} \zeta \right) \hat{\mathbf{e}}_z. \end{aligned}$$

Consequently,

$$\begin{aligned} \mu_0 \mathbf{j} \times \mathbf{B}_1 &= \left[\frac{dB_z}{dr} \frac{1}{r} \frac{d(r\xi B_\theta)}{dr} - \left(k_z \frac{1}{r} \frac{d(rB_\theta)}{dr} - \frac{m}{r} \frac{dB_z}{dr} \right) \zeta + \frac{1}{r} \frac{d(rB_\theta)}{dr} \frac{d(\xi B_\theta)}{dr} \right] \hat{\mathbf{e}}_r \\ &\quad + i \left(\frac{m}{r} B_\theta + k_z B_z \right) \xi \left(\frac{1}{r} \frac{d(rB_\theta)}{dr} \hat{\mathbf{e}}_\theta + \frac{dB_z}{dr} \hat{\mathbf{e}}_z \right) \end{aligned}$$

since $\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = -\frac{dB_z}{dr} \hat{\mathbf{e}}_\theta + \frac{1}{r} \frac{d(rB_\theta)}{dr} \hat{\mathbf{e}}_z$, whence

$$\begin{aligned} \mu_0 \boldsymbol{\xi}^* \cdot \mathbf{j} \times \mathbf{B}_1 &= \frac{dB_z}{dr} \frac{1}{r} \frac{d(r\xi B_\theta)}{dr} \xi^* - \left[k_z \frac{1}{r} \frac{d(rB_\theta)}{dr} - \frac{m}{r} \frac{dB_z}{dr} \right] \xi^* \zeta \\ &\quad + \frac{1}{r} \frac{d(rB_\theta)}{dr} \frac{d(\xi B_\theta)}{dr} \xi^* \\ &\quad + \xi \left[\frac{1}{r} \frac{d(rB_\theta)}{dr} (k_z \zeta^* + B_\theta \eta^*) - \left(\frac{m}{r} \zeta^* - B_z \eta^* \right) \right] \end{aligned}$$

upon invoking $\xi_\theta = -i \frac{k_z \zeta + B_\theta \eta}{(m/r) B_\theta + k_z B_z}$ and $\xi_z = i \frac{(m/r) \zeta - B_z \eta}{(m/r) B_\theta + k_z B_z}$.

We also have

$$(\nabla \cdot \boldsymbol{\xi}^*) \xi \cdot \nabla p = -\mu_0^{-1} \left(\frac{1}{r} \frac{d(r\xi^*)}{dr} + \eta^* \right) \xi \left(\frac{B_\theta}{r} \frac{d(rB_\theta)}{dr} + B_z \frac{dB_z}{dr} \right),$$

and consequently the integrand in the volume integral in (6.44) is μ_0^{-1} times

$$\begin{aligned} &\left[\left(\frac{m}{r} B_\theta + k_z B_z \right)^2 |\xi|^2 + \left| k_z \zeta - \frac{d(\xi B_\theta)}{dr} \right|^2 + \left| \frac{1}{r} \frac{d(r\xi B_z)}{dr} + \frac{m}{r} \zeta \right|^2 \right] \\ &- \xi \left[B_z \frac{dB_z}{dr} + B_\theta \frac{1}{r} \frac{d(rB_\theta)}{dr} \right] \frac{1}{r} \frac{d(r\xi^*)}{dr} - \frac{1}{r} \frac{d(rB_\theta)}{dr} \frac{d(\xi B_\theta)}{dr} \xi^* \\ &- \frac{dB_z}{dr} \frac{1}{r} \frac{d(r\xi B_z)}{dr} \xi^* + \left[k_z \frac{1}{r} \frac{d(rB_\theta)}{dr} - \frac{m}{r} \frac{dB_z}{dr} \right] (\xi^* \zeta + \xi \zeta^*) + \gamma p \left| \eta + \frac{1}{r} \frac{d(r\xi)}{dr} \right|^2, \end{aligned}$$

from which (6.54) follows on introducing the relevant notation.

Section 6.8

A1) (a) The linearised perturbation equations at $r = R$ may be written

$$\begin{aligned}\sigma(R) \left(\ddot{\xi}_r - 2\Omega \dot{\xi}_\theta + \xi_r R \frac{d\Omega^2}{dR} \right) &= 0, \\ \sigma(R) \left(\ddot{\xi}_\theta + 2\Omega \dot{\xi}_r \right) &= 0,\end{aligned}$$

since $GM/\rho^2 \simeq GM/R^2 = R\Omega^2(R)$.

(b) On assuming the usual time dependence $\exp(-i\omega t)$ and setting the right-hand sides zero ($B_z = 0$), these perturbation equations are

$$\begin{aligned}\left(\omega^2 - R \frac{d\Omega^2}{dR} \right) \xi_r - 2i\omega\Omega \xi_\theta &= 0 \\ 2i\omega\Omega \xi_r + \omega^2 \xi_\theta &= 0,\end{aligned}$$

hence $\omega^2 = 4\Omega^2 + R d\Omega^2/dR = \kappa^2$ (rejecting $\omega^2 = 0$).

(c)

$$\frac{d(r^2\Omega(r))^2}{dr} = 2r^2\Omega(r) \left(2r\Omega + r^2 \frac{d\Omega}{dr} \right) \Rightarrow \frac{1}{r^3} \frac{d(r^2\Omega(r))^2}{dr} = \kappa^2,$$

which identifies a more compact form of (6.109), so $r^2\Omega(r)$ monotonically increasing with r implies $\omega^2 = \kappa^2 > 0$ (i.e. the disc is hydrodynamically stable).

Section 6.11

A1) As in Sect.3.2, from the commutation result

$$\rho \frac{df}{dt} = \frac{\partial}{\partial t}(\rho f) + \nabla \cdot (\rho \mathbf{v} f)$$

involving the material derivative $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ we have

$$\rho \frac{d}{dt} \left(\frac{1}{2} v^2 \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 \right) + \nabla \cdot \left(\frac{1}{2} \rho v^2 \mathbf{v} \right).$$

In addition, $\mathbf{v} \cdot \nabla p = \nabla \cdot (p\mathbf{v}) - p \nabla \cdot \mathbf{v}$, $\mathbf{v} \cdot \nabla \cdot \mathbf{t}_{\parallel} = \nabla \cdot (\mathbf{t}_{\parallel} \cdot \mathbf{v}) - \mathbf{t}_{\parallel} : \nabla \mathbf{v}$, and

$$\begin{aligned} \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} &= -\mathbf{v} \times \mathbf{B} \cdot \mathbf{j} = -\mu_0^{-1}(\mathbf{v} \times \mathbf{B}) \cdot \nabla \times \mathbf{B} \\ &= -\mu_0^{-1} \nabla \cdot [\mathbf{B} \times (\mathbf{v} \times \mathbf{B})] - \mu_0^{-1} \mathbf{B} \cdot \nabla \times (\mathbf{v} \times \mathbf{B}) \\ &= -\mu_0^{-1} \nabla \cdot [\mathbf{B} \times (\mathbf{v} \times \mathbf{B})] - \mu_0^{-1} \mathbf{B} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} + \eta \nabla \times \nabla \times \mathbf{B} \right) \\ &= -\frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0} \right) - \nabla \cdot [\mu_0^{-1} \mathbf{B} \times (\mathbf{v} \times \mathbf{B})] - \nabla \cdot (\eta \mathbf{j} \times \mathbf{B}) - \mu_0 \eta j^2, \end{aligned}$$

on introducing the magnetic induction equation (6.160) where the resistivity η is assumed constant (resistivity gradients are omitted). The viscoresistive energy transport equation in the given form (4.161) is then immediately obtained by dotting the equation of motion (6.159) with $\mathbf{v}$.

However, the incompressibility constraint $\nabla \cdot \mathbf{v} = 0$ implicit in (4.161) need not be invoked, and we can derive an alternative form for the energy transport equation by replacing the term $\nabla \cdot \mu_0^{-1} [\mathbf{B} \times (\mathbf{v} \times \mathbf{B})]$ on noting that $\mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = B^2 \mathbf{v} - \mathbf{B} \mathbf{B} \cdot \mathbf{v}$ and writing $\mathbf{B} \cdot \nabla \mathbf{B} \cdot \mathbf{v} = B^2 \kappa \cdot \mathbf{v} + \mathbf{v} \cdot \nabla_{\parallel} (B^2/2)$ where $\nabla_{\parallel} = \hat{\mathbf{b}} \hat{\mathbf{b}} \cdot \nabla$ from (5.31) — viz.

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left[\left(\frac{1}{2} \rho v^2 + \frac{B^2}{\mu_0} \right) \mathbf{v} \right] \\ - \mathbf{p} : \nabla \mathbf{v} - \mu_0^{-1} B^2 \kappa \cdot \mathbf{v} - \mathbf{v} \cdot \nabla_{\parallel} (B^2/2\mu_0) \\ = -\nabla \cdot (\mathbf{p} \cdot \mathbf{v}) - \nabla \cdot (\eta \mathbf{j} \times \mathbf{B}) - \mu_0 \eta j^2, \end{aligned}$$

which is also applicable to compressible plasma and obviously reduces to (3.7) without the gravity term in the absence of a magnetic field and any dissipation. Further, although parallel viscosity is specified in (6.159), the pressure tensor $\mathbf{p} = p\mathbf{I} + \mathbf{t}$ may be more general in the equation of motion and consequently in this form of the viscoresistive energy transport equation.

A2) Recalling the velocity field

$$\mathbf{v} = (\partial\psi/\partial y) \hat{\mathbf{i}} - (\partial\psi/\partial x) \hat{\mathbf{j}} = (xf'(y) + g'(y)) \hat{\mathbf{i}} - f(y) \hat{\mathbf{j}},$$

we obtain $\nabla \mathbf{v} = (\hat{\mathbf{i}} \partial/\partial x + \hat{\mathbf{j}} \partial/\partial y) \mathbf{v} = f'(y) \hat{\mathbf{i}} \hat{\mathbf{i}} + x(f''(y) + g''(y)) \hat{\mathbf{j}} \hat{\mathbf{i}} - f'(y) \hat{\mathbf{j}} \hat{\mathbf{j}}$, so contracting $\nabla \cdot \mathbf{v} = f'(y) - f'(y) = 0$; and for $\mathbf{s} = \{\nabla \mathbf{v}\}$ and $\mathbf{B} = B(y) \hat{\mathbf{i}}$, $\mathbf{t}_{\parallel} = -\mu \nabla \mathbf{v} : \hat{\mathbf{i}} \hat{\mathbf{i}} [3\hat{\mathbf{i}} \hat{\mathbf{i}} - (\hat{\mathbf{i}} \hat{\mathbf{i}} + \hat{\mathbf{j}} \hat{\mathbf{j}} + \hat{\mathbf{k}} \hat{\mathbf{k}})] = -\mu f'(y) [2\hat{\mathbf{i}} \hat{\mathbf{i}} - (\hat{\mathbf{j}} \hat{\mathbf{j}} + \hat{\mathbf{k}} \hat{\mathbf{k}})]$ so that

$$\nabla \cdot \mathbf{t}_{\parallel} = (\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y}) \cdot [-\mu f'(y) (2\hat{\mathbf{i}} \hat{\mathbf{i}} - \hat{\mathbf{j}} \hat{\mathbf{j}} - \hat{\mathbf{k}} \hat{\mathbf{k}})] = \mu \hat{\mathbf{j}} \frac{\partial}{\partial y} f'(y) = \mu \nabla f'(y).$$

The curl of (6.159) is then simply $\nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) = 0$ — i.e. the curl of

$$\begin{aligned} [(xf'(y) + g'(y)) \hat{\mathbf{i}} - f(y) \hat{\mathbf{j}}] \cdot [f'(y) \hat{\mathbf{i}} \hat{\mathbf{i}} + x(f''(y) + g''(y)) \hat{\mathbf{j}} \hat{\mathbf{i}} - f'(y) \hat{\mathbf{j}} \hat{\mathbf{j}}] \\ = [x(f'(y))^2 + g'(y)f'(y) - xf(y)f''(y) - f(y)g''(y)] \hat{\mathbf{i}} + f(y)f'(y) \hat{\mathbf{j}}, \end{aligned}$$

such that we obtain (6.162) — i.e. $f f''' - f' f'' = 0$ and $f g''' - f'' g' = 0$. The first of these equations may be rewritten $f'''/f'' = f'/f$, which integrates to give $\ln f'' = \ln f + \ln(\pm \lambda^2)$ or $f'' \mp \lambda^2 f = 0$ where λ is a constant, with respective hyperbolic (or exponential) and trigonometric function solutions.

Incidentally, since $\mathbf{t}_{\parallel} = -\mu f'(y) [2\hat{\mathbf{i}}\hat{\mathbf{i}} - (\hat{\mathbf{j}}\hat{\mathbf{j}} + \hat{\mathbf{k}}\hat{\mathbf{k}})]$ we also immediately obtain $\mathbf{t}_{\parallel} : \nabla \mathbf{v} = -3\mu [f'(y)]^2$ — cf. (6.163).

Section 6.12

A1) For $\mathbf{B}_0 = \text{constant}$ and constant T , elimination between the second and third perturbation equations of (6.173) yields

$$\frac{d}{dz}(nv_{1x}) - iknv_{1z} = -\frac{gk}{\omega}n_1,$$

and hence

$$\frac{d}{dz}(nv_{1x}) - iknv_{1z} = i\frac{gk}{\omega^2}\left(\frac{d}{dz}(nv_{1z}) + iknv_{1x}\right)$$

from the first perturbation equation; and setting $B_{1y} = 0$ in the fourth perturbation equation provides the second equation specified in (Q1), which on introducing $\omega_{ci} = eB/m_i$ may be rewritten

$$\frac{dV}{dz} + ikv_{1x} - \frac{\omega}{\omega_{ci}}kv_{1z} = 0,$$

in the form to be invoked in the subsequent derivation. If $(\omega/\omega_{ci})kv_{1z}$ from this form is used in the first term inside the large bracket on the right-hand side of the above equation, on rearranging $-i\omega^2/\omega_{ci}$ times the result we have

$$\begin{aligned} \frac{gk}{\omega_{ci}}\frac{d}{dz}\left(n\frac{dV}{dz}\right) &= -k\left(\frac{\omega^2}{\omega_{ci}} + \frac{gk}{\omega}\right)\frac{dn}{dz}\left(i\frac{\omega}{\omega_{ci}}v_{1x}\right) \\ &\quad - k\left(\frac{\omega^2}{\omega_{ci}} + \frac{gk}{\omega_{ci}}\right)n\left(i\frac{\omega}{\omega_{ci}}\frac{dv_{1x}}{dz}\right) \\ &\quad - k^2\frac{\omega^3}{\omega_{ci}^2}nv_{1z} - \frac{gk^3}{\omega_{ci}}n\left(i\frac{\omega}{\omega_{ci}}v_{1x}\right), \end{aligned}$$

where $V = v_{1z} - i(\omega/\omega_{ci})v_{1x}$. Then adding the derivative of ωn times the rewritten form of the second equation, after the cancellation of some terms we obtain

$$\left(\omega + \frac{gk}{\omega_{ci}}\right)\frac{d}{dz}\left(n\frac{dV}{dz}\right) = k\left(\frac{\omega^2}{\omega_{ci}} + \frac{gk}{\omega}\right)V\frac{dn}{dz} + \left(\omega + \frac{gk}{\omega_{ci}}\right)k^2nV,$$

which is (6.174) when rearranged.